UFP on the Line: LP Relaxations and Integrality Gaps

Chandra Chekuri

Univ of Illinois, Urbana-Champaign

Based on paper with Alina Ene and Nitish Korula

See longer version of APPROX'09 paper on web page of Chandra or Alina

Problem

- Line on n nodes viewed as a graph
- Each edge e has capacity $u(e)$
- m interval demands (s_i, t_i, w_i, d_i)

$\max \sum_i w_i x_i$

$\sum_{e \in [si,ti]} d_i x_i \leq u(e)$ for all e

$x_i \in \{0,1\}$ for all i

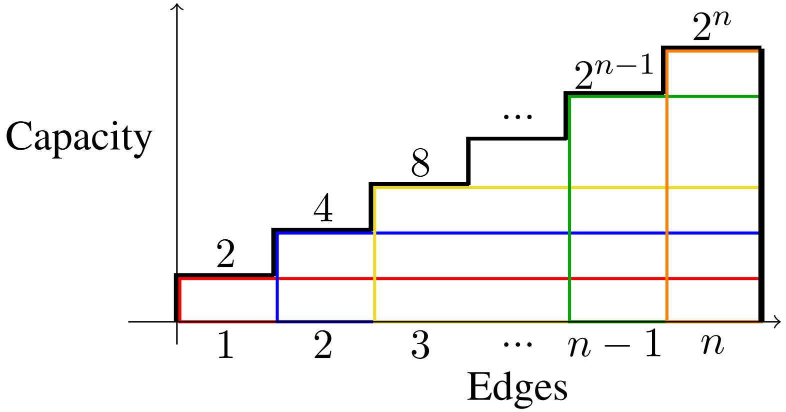
Basic-LP

$\max \sum_i w_i x_i$

$\sum_{e \in [si,ti]} d_i x_i \leq u(e)$ for all e

$x_i \in [0,1]$ for all i

Basic-LP has integrality gap $\Omega(n)$



Theorem: Integrality gap of Basic-LP is $O(1/\delta^4)$ if $d_i \leq (1-\delta)b_i$ for all i

Focus on large demands: $d_i > \frac{3}{4} b_i$

There is an $O(1)$ approximation for large demands via dynamic programming [Bonsma-Schulz-Wiese'01,AGLW'13]

Quest: Is there a "natural" LP with $O(1)$ gap?

$B(e)$: large demands with e on their path

Rank-LP [CEK'09]

$\max \sum_i w_i\, x_i$

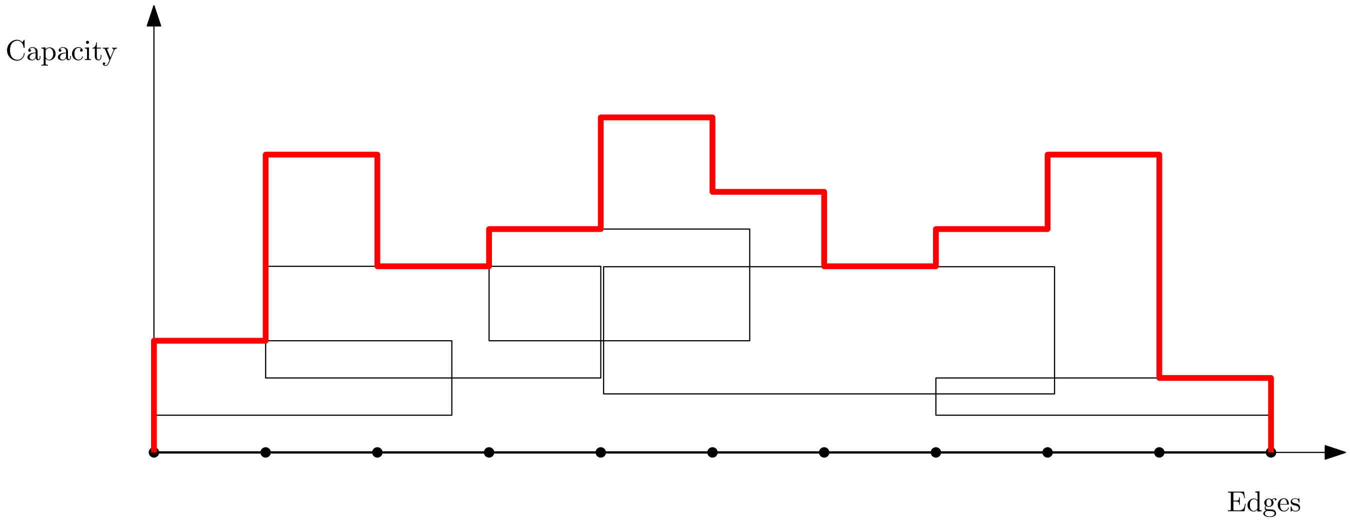
$\sum_{e \in [si,ti]} d_i\, x_i \leq u(e)$ for all e

$\sum_{i \in S} x_i \leq f(S)$ for all e, $S \subseteq B(e)$

$x_i \in [0,1]$ for all i

Compact UFP-LP $\max \sum_i w_i x_i$

$$
\begin{array}{rcll}
\sum_{i:\, e \in P_i} d_i x_i & \leq & c_e & (\forall e \in E(G)) \\
\sum_{R_j \in \mathrm{LeftBlock}(e,i)} x_j & \leq & 1 & (\forall e \in E(G), R_i \in \mathcal{B}_{left}(e)) \\
\sum_{R_j \in \mathrm{RightBlock}(e,i)} x_j & \leq & 1 & (\forall e \in E(G), R_i \in \mathcal{B}_{right}(e)) \\
x_i & \in & [0,1] & (\forall i \in \{1, \ldots, k\})
\end{array}
$$


Capacity
Edges

B(e) : demands with e on their path

Top-Drawn-Rectangle-LP [AGLW'13]

$\max \sum_i w_i x_i$

$\sum_{i:\, p \in Rect(i)} x_i \leq 4$ for all p

$x_i \in [0,1]$ for all i

Theorem: Integrality gap of Rank-LP is O(log n)

Theorem: [AGLW'13] Integrality gap of Top-Drawn-Rectanlge LP is O(1) for *unweighted* instances

Theorem: Integrality gap of Rank-LP is O(α) where α is integrality gap of Top-Drawn-Rectangle LP

Question: Is the integrality gap of Rank-LP O(1)?

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Question: Is the integrality gap of Rank-LP O(1)?

Why do I care?
Could perhaps extend to submodular function maximization.

UPF on Trees: $O(\log^2 n)$ combinatorial approximation

Is there a better approximation?

LP Relaxation?