Existence of outer automorphisms of the Calkin algebra is independent of ZFC: One half of the proof

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- With $A = \overline{\bigcup_{n=0}^{\infty} A_n}$, an automorphism $\varphi \in \mathrm{Aut}(A)$ determined by $\varphi(a) = v_n a v_n^*$ for $a \in A_n$.

It probably does not follow that φ is implemented by a unitary, not even (as will happen in our case) if $v_n \in A_{n+1}$ for all $n \in \mathbb{Z}_{\geq 0}$.

Stated without extraneous structure: If $A \subset Q$ is a separable subalgebra and $\varphi \in \mathrm{Aut}(A)$ is approximately inner in A, it probably does not follow that φ is implemented by a unitary in Q.

However, it is true that if $\varphi \in \mathrm{Aut}(A)$ is asymptotically inner in A (definition: next slide), then φ is implemented by a unitary in Q.

Asymptotic innerness

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Let A be a unital C*-algebra, and let $\varphi, \psi \in \mathrm{Aut}(A)$. Then φ is *asymptotically unitarily equivalent* to ψ if there is a continuous path $t \mapsto u_t$, for $t \in [0, \infty)$, in the unitary group of A such that $\lim_{t \to \infty} u_t \varphi(a) u_t^* = \psi(a)$ for all $a \in A$.

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If $\beta < 2^{\aleph_0}$ is a limit ordinal, we will take $A = \overline{\bigcup_{\gamma<\beta} A_\gamma}$ and let $\varphi \in \mathrm{Aut}(A)$ be the automorphism determined by $\varphi|_{A_\gamma} = \varphi_\gamma$ whenever $\gamma < \beta$. We need the automorphisms φ_γ to be implemented by unitaries $v_\gamma \in Q$.

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Set $p_n = e_{n+1} - e_n$ for $n \in \mathbb{Z}_{\geq 0}$. We have $\sum_{n=0}^{\infty} p_n = 1$, $\lim_{n\to\infty} \|\pi(w_n)a\pi(w_n)^* - \varphi(a)\| = 0$ for all $a \in A$, $\lim_{n\to\infty}(p_n b - b p_n) = 0$ for all $b \in B$, and $\lim_{n\to\infty}(p_n w_n - w_n p_n) = 0$. Define

$$w = \sum_{n=0}^{\infty} p_n w_n p_n.$$

Then $\pi(w)$ is unitary.

Let $b \in B$. Then $b - \sum_{n=0}^{\infty} p_n b p_n$ is compact, and one can check that

$$w \left(\sum_{n=0}^{\infty} p_n b p_n \right) w^* - \sum_{n=0}^{\infty} p_n w_n b w_n^* p_n$$

is compact. Moreover, one can check that

$$\pi \left(\sum_{n=0}^{\infty} p_n w_n b w_n^* p_n \right) = \lim_{n\to\infty} \pi(w_n)\pi(b)\pi(w_n)^* = \varphi(\pi(b)).$$

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