Hecke operators

Frank's grad workshop @ Fields 3/12/2012

Recall: MF (level 1)

① $f: \mathcal{H} \to \mathbb{C}$, $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ + holo + growth

$\lambda \in \mathbb{C}^\times$

② $f: \{\text{lattices in } \mathbb{C}\} \to \mathbb{C}$, $f(\lambda\Lambda) = \lambda^{-k} f(\Lambda)$. — " —

③ $f: \{(E,\omega)\} \to \mathbb{C}$, $f(E, \lambda\omega) = \lambda^{-k} f(E,\omega)$ — " —

ell. cv. $\quad \in H^0(E, \Omega^1) - \{0\}$. corrected ...

i.e. $f \in H^0(X(1), \underline{\omega}^{\otimes k})$

namely: value at $E \in Y(1)$ is $f(E,\omega)\,\omega^{\otimes k}$ (indep. of choice of $\omega \neq 0$!)

Hecke ops.:

② $n \geq 1$:

$$(T_n f)(\Lambda) := \sum_{(\Lambda:\Lambda')=n} f(\Lambda')$$

classically put $\frac{1}{n}$

clearly modular.

① $n = p$ prime

$$\Lambda = \mathbb{Z} + \mathbb{Z}\tau$$

$$\Lambda' = \begin{cases} \mathbb{Z} + \mathbb{Z}\frac{\tau+l}{p} & 0 \leq l \leq p-1 \\ \frac{1}{p}\mathbb{Z} + \mathbb{Z}\tau \end{cases}$$

$$\rightsquigarrow (T_p f)(\tau) = \sum_{0}^{p-1} f\left(\frac{\tau + l}{p}\right) + p^k f(p\tau)$$

(3) lattice $\Lambda \longleftrightarrow (E = \mathbb{C}/\Lambda,\ \omega = dz)$

(Λ, Λ') index p $\longleftrightarrow$ $(E' = \mathbb{C}/\Lambda' \xrightarrow[\deg. p]{} E = \mathbb{C}/\Lambda\ \cancel{\deg. p},\ \omega = dz)$ both!

$$Y_0(p) = \{(E, E', \alpha) : \alpha : E' \longrightarrow E \ \deg p\}/\cong$$

$$\begin{tikzcd} & Y_0(p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y(1) & & Y(1) \end{tikzcd}$$

$f \in H^0(X(1), \underline{\omega}^{\otimes k})$

$\rightsquigarrow T_p f = \pi_{1*} \pi_2^* f$ (extends to cusps)

$\uparrow$ sum over fibres.

Hecke alg. $\mathbb{C}[T_n : n \geq 1]$ comm.

eigenform $f = \sum a_n q^n \rightsquigarrow T_n f = n a_n f$.

MFs $\rightsquigarrow$ $GL_2(\mathbb{A}_f)$ reps.

$N \geq 1$

N-torsion in $\mathbb{C}/\Lambda$.

$$\mathrm{Latt}(N) := \left\{ (\Lambda, j) : \Lambda \text{ lattice in } \mathbb{C},\ j : (\mathbb{Z}/N)^2 \xrightarrow[\text{level}]{\sim} \tfrac{1}{N}\Lambda/\Lambda \right\}$$

$$M(N) := \left\{ f : \mathrm{Latt}(N) \to \mathbb{C} : f(\lambda\Lambda, \lambda j) = \lambda^{-k} f(\Lambda, j)\ \forall \lambda \in \mathbb{C}^\times, \text{ + holo + growth} \right\}.$$

$\mathbb{C} \cong \mathbb{R}^2$ basis $i, 1$

$$\begin{array}{ccc} & \mathrm{Latt}(N) & \\ \circlearrowleft & & \circlearrowleft \\ GL_2(\mathbb{R}) & & GL_2(\mathbb{Z}/N) \\ \Downarrow & & \Downarrow \\ \alpha & & \beta \end{array}$$

$$\alpha(\Lambda, j) = ({}^t\alpha(\Lambda),\ {}^t\alpha \circ j)$$
$$\beta(\Lambda, j) = (\Lambda,\ j \circ {}^t\beta^{-1})$$

comm. right actions.

Base pt. $(\mathbb{Z} + \mathbb{Z}i, (\frac{i}{N}, \frac{1}{N}))$

Stab $= GL_2(\mathbb{Z})$

$$\Rightarrow \mathrm{Latt}(N) = \underset{\text{(diag.)}}{GL_2(\mathbb{Z})} \backslash GL_2(\mathbb{R}) \times GL_2(\mathbb{Z}/N) \qquad (*_N)$$

θ

$\mathcal{H}$

$$\underset{u+iv}{\overset{\shortparallel}{\tau}} \longmapsto (\mathbb{Z} + \mathbb{Z}\tau, (\tfrac{\tau}{N}, \tfrac{1}{N})) \longmapsto \left[\begin{pmatrix} v & u \\ & 1 \end{pmatrix}, 1 \right].$$

induces

$$\Gamma(N)\backslash\mathcal{H} \hookrightarrow \underbrace{\underset{\sim\sim}{\overset{\text{ell. cv. + level } N}{\mathrm{Latt}(N)}}/\mathbb{C}^\times \xrightarrow{\sim} GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})\times GL_2(\mathbb{Z}/N)/\underset{\mathbb{R}^\times SO_2(\mathbb{R})}{\overset{\|}{\mathbb{C}^\times}}}_{\circlearrowleft\; GL_2(\mathbb{Z}/N)}$$

$$\mathrm{RHS} = \coprod_{\gamma\in\frac{GL_2(\mathbb{Z}/N)}{SL_2(\mathbb{Z}/N)} = (\mathbb{Z}/N)^\times} \gamma(\Gamma(N)\backslash\mathcal{H})$$

(Weil pairing).

So $M(N) \underset{\text{non-can.}}{\longleftrightarrow} \varphi(N) = |(\mathbb{Z}/N)^\times|$ classical MFs of level $\Gamma(N)$.

<u>Vary N</u>:

If $M|N$ have nat. maps $(*_N) \longrightarrow (*_M)$:

$$\begin{array}{ccccc}
\mathcal{H} & \hookrightarrow & \mathrm{Latt}(N) & \longrightarrow & \cdots \\
\| & & \big\downarrow \text{forget} & & \big\downarrow \text{nat} \\
\mathcal{H} & \hookrightarrow & \mathrm{Latt}(M) & \longrightarrow & \cdots
\end{array}$$

$N = p^m$.

$M(p^\infty) := \{ f : GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})\times GL_2(\mathbb{Z}_p) \longrightarrow \mathbb{C}$:

$\cdot\ f(g\lambda) = \lambda^{-k} f(g) \quad \forall \lambda\in\mathbb{C}^\times = \mathbb{R}^\times SO_2(\mathbb{R})$

- $\exists m \geq 1$ s.t.

$$f(gu) = f(g) \quad \forall u \in K(m) := \ker\left(GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{Z}/p^m)\right)$$

- holo + growth $\}$.

$$M(p^\infty) = \bigcup_m \underbrace{M(p^\infty)^{K(m)}}_{= M(p^m)}$$

$\circlearrowleft$

$GL_2(\mathbb{Z}_p)$

[<u>Lemma:</u> $GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R}) \times GL_2(\mathbb{Z}_p) \xrightarrow{\sim} GL_2(\mathbb{Z}[\tfrac{1}{p}]) \backslash GL_2(\mathbb{R}) \times GL_2(\mathbb{Q}_p)$

<u>Pf:</u> surj.: $GL_2(\mathbb{Z}[\tfrac{1}{p}])\, GL_2(\mathbb{Z}_p) = GL_2(\mathbb{Q}_p)$ as $\mathbb{Z}[\tfrac{1}{p}]$ dense in $\mathbb{Q}_p$.

inj: $GL_2(\mathbb{Z}[\tfrac{1}{p}]) \cap GL_2(\mathbb{Z}_p) = GL_2(\mathbb{Z})$. $\square$

$\Rightarrow GL_2(\mathbb{Q}_p)$-action on $M(p^\infty)$!

<u>Ex.:</u> [Action of $\begin{pmatrix} 1 & \\ & p \end{pmatrix}$]

$f \in M(1)$ classical, level 1 MF.

$\begin{pmatrix} 1 & \\ & p \end{pmatrix} f \in M(p)$

$\begin{pmatrix} 1 & \\ & p \end{pmatrix} f \circ \theta$ classical, level $\Gamma(p)$ (even $\Gamma_0(p)$)

$$\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix} f \circ \theta\right)(\tau) = \left(\begin{pmatrix} 1 & \\ & p \end{pmatrix} f\right)\left(\begin{pmatrix} v & u \\ & 1 \end{pmatrix}, 1\right)$$
$$= f\left(\begin{pmatrix} v & u \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & p \end{pmatrix}\right)$$

$= f\left(\begin{pmatrix}1 & \\ & p\end{pmatrix}^{-1}\begin{pmatrix}v & u\\ & 1\end{pmatrix}, 1\right)$ by $GL_2(\mathbb{Z}[\frac{1}{p}])$-inv.

$= \cancel{}\, p^{-k} f\left(\begin{pmatrix}pv & pu\\ & 1\end{pmatrix}, 1\right)$ by scaling. $[p \in \mathbb{C}^\times]$

$= p^{-k} f(pt)$ <u>degeneracy map</u>!

[If time: $GL_2(\mathbb{Q}_p) = \bigcup_{r \leq s} \underbrace{GL_2(\mathbb{Z}_p)\begin{pmatrix}p^r & \\ & p^s\end{pmatrix} GL_2(\mathbb{Z}_p)}_{\text{deg. maps}}$]

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f new form level 1 (prime to p) $\rightsquigarrow$ $\pi_{f,p} :=$ $GL_2(\mathbb{Q}_p)$-rep. gen^d by f in $M(p^\infty)$

- $\pi_{f,p} = \bigcup_m \pi_{f,p}^{K(m)}$ "smooth". (cpt. open subgps.)

[<u>Thm.</u> ("Atkin-Lehner") $\pi_{f,p}$ irred.

Suppose π irred. smooth $GL_2(\mathbb{Q}_p)$-rep.

$K := GL_2(\mathbb{Z}_p)$

$\mathcal{H}_p := \mathbb{C}[K \backslash GL_2(\mathbb{Q}_p)/K]$ acts on π^K (Hecke)

$[KgK]f = \sum g_\alpha f$, where $KgK = \coprod_{\text{fin.}} g_\alpha K$

[<u>Thm</u> (Satake)

$\mathcal{H}_p = \underset{\text{comm.}}{\mathbb{C}[T_p, S_p^{\pm 1}]}$, where $T_p = [K\begin{pmatrix}p & \\ & 1\end{pmatrix}K]$

$S_p = [K\begin{pmatrix}p & \\ & p\end{pmatrix}K]$

Thm: If $\pi^K \neq 0$, then $\dim \pi^K = 1$ and π is det^d by $\mathcal{H}_p$-evals on π^K.

$$\pi_{f,p}^K = \mathbb{C}\cdot f$$

$$T_p f = p\,\alpha_p f \qquad \text{(prev. one!)}$$

$$S_p f = p^k f$$

$\Rightarrow a_p$ (+ wt.) determines $\pi_{f,p}$.

Local Langlands:

$$\begin{tikzcd}[column sep=large]
\boxed{\begin{array}{c}\text{irr. } GL_2(\mathbb{Q}_p)\text{-repr. } \pi \\ \pi^K \neq 0\end{array}} \arrow[r, leftrightarrow, "\text{unram. LL}"] \arrow[d, leftrightarrow] & \boxed{\begin{array}{c}\text{unram. Gal. rep.} \\ \text{“}Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow GL_2(\mathbb{C})\text{”} \\ \mathrm{Frob} \longmapsto \begin{pmatrix}\alpha_p & \\ & \beta_p\end{pmatrix}\end{array}} \arrow[d, leftrightarrow] \\
\text{Hecke evals } \mathcal{H}_p \to \mathbb{C} \arrow[r, leftrightarrow, bend right, "\substack{\text{canonical} \\ \text{Satake}}"'] & \begin{array}{c}\text{conj. class } \begin{pmatrix}\alpha_p & \\ & \beta_p\end{pmatrix} \in GL_2(\mathbb{C}) \\ \text{char. poly. } x^2 - a_p x + p^{k-1}.\end{array}
\end{tikzcd}$$

general LL: drop “$\pi^K \neq 0$” + “unram.” (higher level MF)

[**Galois side:** WD group or ℓ-adic Gal. rep. ($\ell \neq p$).]

irred $GL_2(\mathbb{Q}_p)$-reps

unram. $\Gamma(1)$	$\Longrightarrow$	princ. series	$\mathrm{Ind}_{\begin{pmatrix} * & * \\ & * \end{pmatrix}}^{GL_2(\mathbb{Q}_p)} (\chi_1 \otimes \chi_2)$, $\chi_1\chi_2^{-1} \neq \lvert\cdot\rvert^{\pm 1}$	$\overset{LL}{\longleftrightarrow}$	$\chi_1 \oplus \chi_2$. (LCFT!)
$\Gamma_0(p)$	$\Longrightarrow$	special	$St \otimes (\chi \circ \det)$	$\longleftrightarrow$	$\begin{pmatrix} \chi\lvert\cdot\rvert^{\frac{1}{2}} & * \\ & \chi\lvert\cdot\rvert^{-\frac{1}{2}} \end{pmatrix}$
		1-dim*	$\chi \circ \det$	$\longleftrightarrow$	$\begin{pmatrix} \cdot\cdot & 0 \\ & \cdot\cdot \end{pmatrix}$
		supercusp.	everything else!	$\longleftrightarrow$	irred.

not for MF!
(oldforms)