

# Hecke operators

Frank's  
Prad workshop  
@ Fields  
3/12/2012

Recall: MF (level 1)

①  $f: \mathcal{H} \rightarrow \mathbb{C}$ ,  $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$  + holes + growth  $k \in \mathbb{Z}$

②  $f: \{\text{lattices in } \mathbb{C}\} \rightarrow \mathbb{C}$ ,  $f(\lambda\Lambda) = \lambda^{-k} f(\Lambda)$

③  $f: \{(E, w)\} \rightarrow \mathbb{C}$ ,  $f(E, \lambda w) = \lambda^{-k} f(E, w)$  Corrected...

all  $w \in H^0(E, \Omega^1) - \{0\}$

i.e.  $f \in H^0(X(1), \underline{w}^{\otimes k})$

namely: value at  $E \in Y(1)$  is  $f(E, w) w^{\otimes k}$  (indep. of choice of  $w \neq 0$ !)

Hecke ops.:

②  $n \geq 1$ :

$$(T_n f)(\Lambda) := \sum_{\Lambda: \Lambda' = n\Lambda} f(\Lambda')$$

$\uparrow \Lambda: \Lambda' = n\Lambda$

classically put  $\frac{1}{n}$

clearly modular.

①  $n = p$  prime

$$\Lambda = \mathbb{Z} + \mathbb{Z}\tau$$

$$\Lambda' = \begin{cases} \mathbb{Z} + \mathbb{Z} \frac{\tau+l}{p} \\ \frac{1}{p} \mathbb{Z} + \mathbb{Z}\tau \end{cases}$$

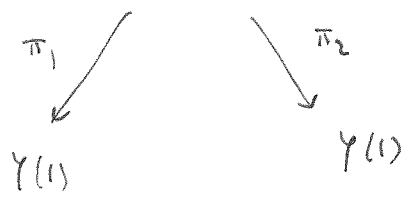
$$0 \leq l \leq p-1$$

$$\rightarrow (T_p f)(\tau) = \sum_0^{p-1} f\left(\frac{\tau+l}{p}\right) + p^k f(p\tau)$$

③ lattice  $\lambda \mapsto (E = \mathbb{C}/\lambda, \omega = dz)$

$(\lambda, \lambda')$   $\leftrightarrow$   $(E' = \mathbb{C}/\lambda' \xrightarrow{\text{deg. } p} E = \mathbb{C}/\lambda \xrightarrow{\text{deg. } p}, \omega = dz)$   
 under  $p$  both!

$$Y_0(p) = \{(E, E', \alpha) : \alpha : E' \rightarrow E \text{ deg } p\} / \cong$$



$$f \in H^0(Y(1), \omega^{\otimes k})$$

$$\rightarrow T_p f = \pi_{1*} \pi_2^* f$$

↑  
sum over fibres.

(extends to cusps)

Hecke alg.  $\mathbb{C}[T_n : n \geq 1]$  comm.

eigenform  $f = \sum a_n \tau^n \rightarrow T_n f = n a_n f.$

MFs  $\rightsquigarrow$   $GL_2(\mathbb{Q})$ -reps.

$N$ -torsion  
in  $\mathbb{C}/\Lambda$ .

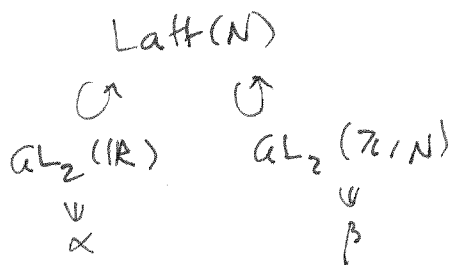
$N \geq 1$

$$\text{Latt}(N) := \{ (\Lambda, j) : \Lambda \text{ lattice in } \mathbb{C}, j: (\mathbb{Z}/N)^2 \xrightarrow{\sim} \frac{1}{N}\Lambda/\Lambda \}$$

level

$$\Pi(N) := \{ f: \text{Latt}(N) \rightarrow \mathbb{C} : f(\lambda\Lambda, \lambda j) = \lambda^{-k} f(\Lambda, j) \forall \lambda \in \mathbb{C}^\times, \text{ + holo + growth } f. \}$$

$\mathbb{C} \cong \mathbb{R}^2$  basis  $i, 1$



$$\alpha(\Lambda, j) = ({}^t \alpha(\Lambda), {}^t \alpha \circ j)$$

$$\beta(\Lambda, j) = (\Lambda, j \circ {}^t \beta^{-1})$$

comm. right actions.

Base pt.  $(\mathbb{Z} + \mathbb{Z}i, (\frac{i}{N}, \frac{1}{N}))$

$$\text{Stab} = GL_2(\mathbb{Z})$$

$$\Rightarrow \text{Latt}(N) = GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R}) \times GL_2(\mathbb{Z}/N)$$

(diag.)



$(*_N)$

$\tau$

$$\tau \longmapsto (\mathbb{Z} + \mathbb{Z}\tau, (\frac{\tau}{N}, \frac{1}{N})) \longmapsto [ \begin{pmatrix} v & u \\ & 1 \end{pmatrix}, 1 ]$$

"  
u+iv

induces

$$\Gamma(N) \backslash \mathcal{H} \hookrightarrow \frac{\text{Latt}(N)}{\mathbb{C}^\times} \xrightarrow{\sim} \underbrace{\text{GL}_2(\mathbb{Z}) \backslash \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{Z}/N) / \mathbb{C}^\times}_{\substack{\uparrow \\ \text{GL}_2(\mathbb{Z}/N)}} \cong \mathbb{R}^\times \text{SO}_2(\mathbb{R})$$

(all. cv. + level N)

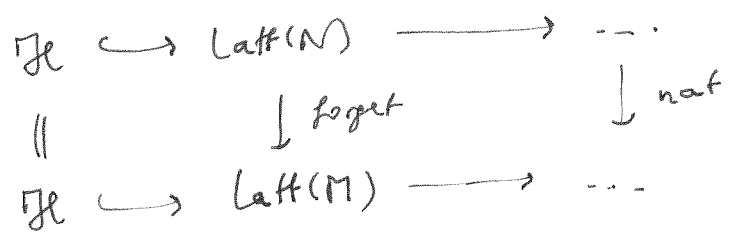
$$\text{RHS} = \coprod_{\gamma \in \text{GL}_2(\mathbb{Z}/N) / \mathbb{Z}(N)^\times} \Gamma(N) \backslash \mathcal{H}$$

(Weil pairing).

So  $M(N) \xleftrightarrow{\text{non-com.}} \varphi(N) = (\mathbb{Z}/N)^\times$  classical MFs of level  $\Gamma(N)$ .

Vary N:

$\Gamma(N)$  have nat. maps  $(*_N) \rightarrow (*_M)$ :



$$N = p^m$$

$$M(p^\infty) := \left\{ f: \text{GL}_2(\mathbb{Z}) \backslash \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{Z}_p) \longrightarrow \mathbb{C} : \right.$$

$$\left. \cdot f(gd) = d^{-k} f(g) \quad \forall d \in \mathbb{C}^\times = \mathbb{R}^\times \text{SO}_2(\mathbb{R}) \right\}$$

•  $\exists m \geq 1$  st.

$$f(gu) = f(g) \quad \forall u \in K(m) := \ker(\mathrm{GL}_2(\mathbb{Z}_p) \rightarrow \mathrm{GL}_2(\mathbb{Z}/p^m))$$

• holo + growth  $f$ .

$$\Gamma(p^\infty) = \bigcup_m \underbrace{\Gamma(p^\infty)^{K(m)}}_{= \Gamma(p^m)}$$

$$\uparrow \\ \mathrm{GL}_2(\mathbb{Z}_p)$$

[Lemma:  $\mathrm{GL}_2(\mathbb{Z}) \backslash \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Z}[\frac{1}{p}]) \backslash \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Q}_p)$

Pf: surj.:  $\mathrm{GL}_2(\mathbb{Z}[\frac{1}{p}]) \mathrm{GL}_2(\mathbb{Z}_p) = \mathrm{GL}_2(\mathbb{Q}_p)$  as  $\mathbb{Z}[\frac{1}{p}]$  dense in  $\mathbb{Q}_p$ .

inj.:  $\mathrm{GL}_2(\mathbb{Z}[\frac{1}{p}]) \cap \mathrm{GL}_2(\mathbb{Z}_p) = \mathrm{GL}_2(\mathbb{Z})$ .  $\square$

$\Rightarrow \mathrm{GL}_2(\mathbb{Q}_p)$ -action on  $\Gamma(p^\infty)$ !

Ex.: [Action of  $(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix})$ ]

$f \in \Gamma(1)$  classical, level 1 MF.

$$\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right) f \in \Gamma(p)$$

$\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right) f \circ \theta$  classical, level  $\Gamma(p)$  (even  $\Gamma_0(p)$ )

$$\begin{aligned} \left(\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right) f \circ \theta\right)(\tau) &= \left(\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right) f\right)\left(\begin{pmatrix} v & u \\ & 1 \end{pmatrix}, 1\right) \\ &= f\left(\begin{pmatrix} v & u \\ & 1 \end{pmatrix}, \left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right)\right) \end{aligned}$$

$$\begin{aligned}
 &= f\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix}^{-1} \begin{pmatrix} v & u \\ & 1 \end{pmatrix}, 1\right) \quad \text{by } GL_2(\mathbb{Z}[1/p])\text{-inv.} \\
 &= p^{-k} f\left(\begin{pmatrix} p^v & p^u \\ & 1 \end{pmatrix}, 1\right) \quad \text{by scaling.} \quad [p \in \mathbb{C}^\times] \\
 &= p^{-k} f(p\tau) \quad \text{degeneracy map!}
 \end{aligned}$$

[if true:  $GL_2(\mathbb{Q}_p) = \bigcup_{\Gamma \leq S} GL_2(\mathbb{Z}_p) \underbrace{\begin{pmatrix} p^\Gamma & \\ & p^\Gamma \end{pmatrix}}_{\text{deg. maps}} GL_2(\mathbb{Z}_p)$  ]

$f$  new form level 1 (prime to  $p$ )  $\rightsquigarrow \pi_{f,p} := GL_2(\mathbb{Q}_p)$ -rep. gen<sup>d</sup> by  $f$  in  $\Pi(p^\infty)$

$\pi_{f,p} = \bigcup_m \pi_{f,p}^{K(m)}$  "smooth". (opt. (open subgps.))

[Thm. ("Atkin-Lehner")  $\pi_{f,p}$  mod.]

Suppose  $\pi$  mod. smooth  $GL_2(\mathbb{Q}_p)$ -rep.

$K := GL_2(\mathbb{Z}_p)$

$\mathcal{H}_p := \mathbb{C}[K \backslash GL_2(\mathbb{Q}_p) / K]$  acts on  $\pi^K$  (Hecke)

$[KgK]f = \sum g_\alpha f$ , where  $KgK = \bigsqcup_{\text{fin.}} g_\alpha K$

[Thm (Satake)

$\mathcal{H}_p = \mathbb{C}[T_p, S_p^{\pm 1}]$ , where  $T_p = [K \begin{pmatrix} p & \\ & 1 \end{pmatrix} K]$   
 $S_p = [K \begin{pmatrix} p & \\ & p \end{pmatrix} K]$   
 comm.

[Thm: If  $\pi^k \neq 0$ , then  $\dim \pi^k = 1$  and  $\pi$  is det<sup>d</sup> by  $\mathcal{H}_p$ -vals on  $\pi^k$ .]

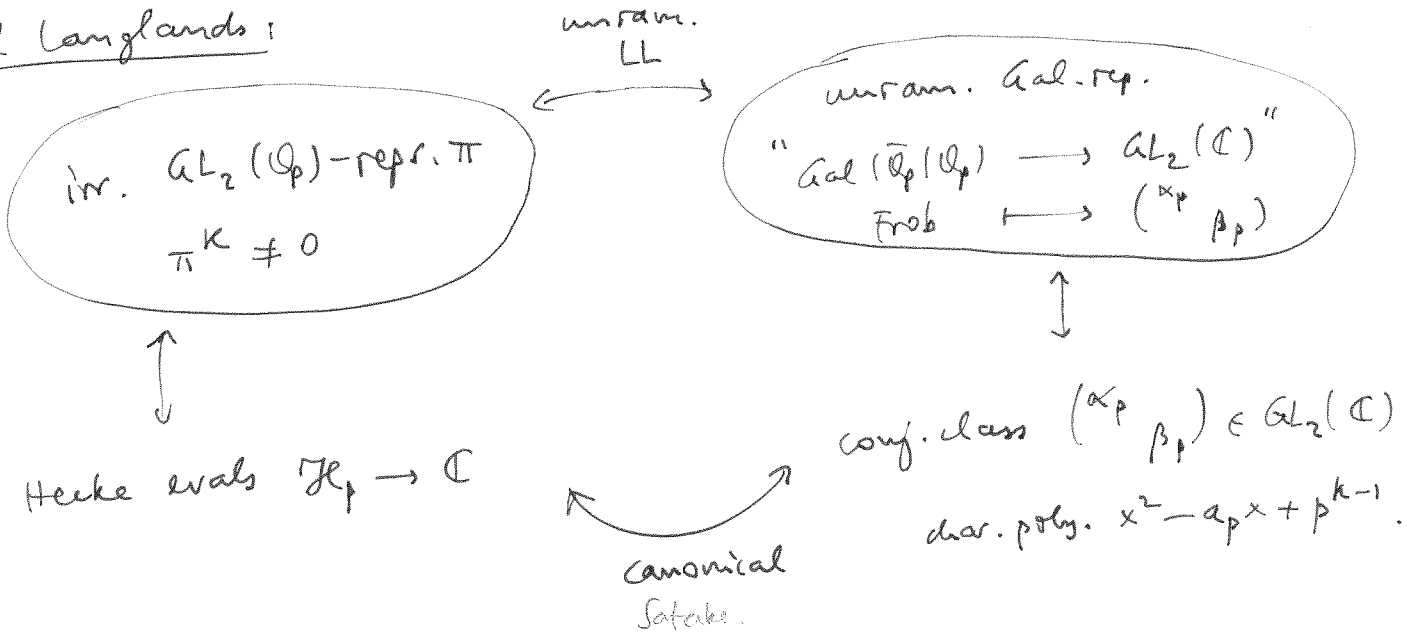
$$\pi_{f,p}^k = \mathbb{C} \cdot f$$

$$T_p f = p a_p f \quad (\text{prev. one!})$$

$$S_p f = p^k f$$

$\Rightarrow a_p$  (+ wt.) determines  $\pi_{f,p}$ .

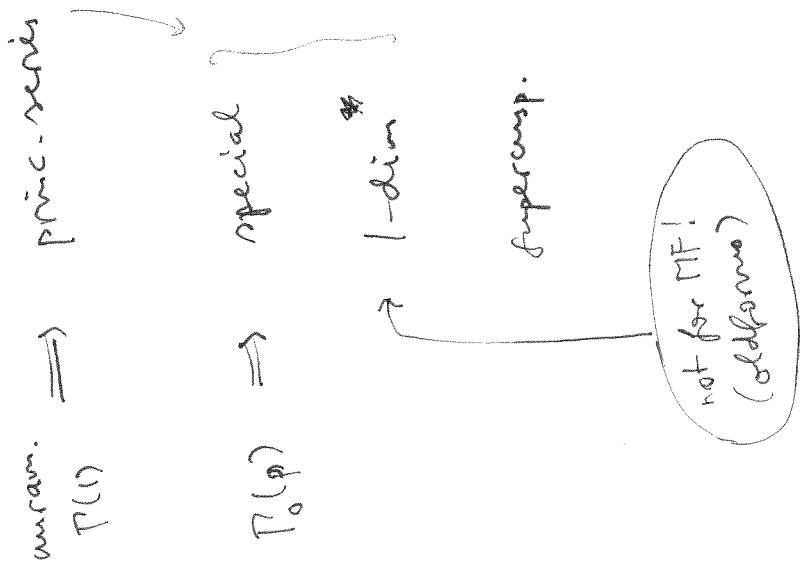
Local Langlands:



general LL: loop "  $\pi^k \neq 0$  " + " unram. " (higher level MF)

[Galois side: WD group or  $l$ -adic Gal. rep. ( $l \neq p$ ).]

ined  $GL_2(\mathbb{Q}_p)$ -repr



$GL_2(\mathbb{Q}_p)$   
 $(\ast \ast)$   
 $(X_1 \otimes X_2)$   
 $X_1, X_2^{-1} \neq 1, 1 \neq 1$

$Sf \otimes (X \circ \det)$

$X \circ \det$

everything else!

