Analysis of Fourier Transform Valuation Formulas and Applications

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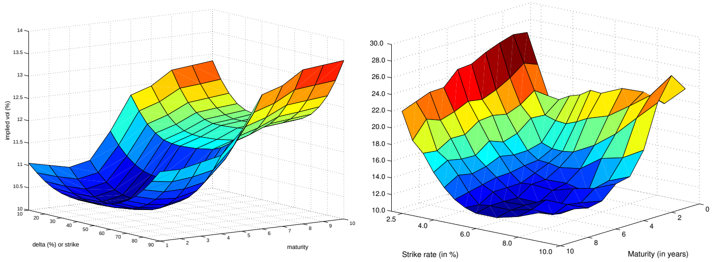
Volatility surface



Volatility surfaces of foreign exchange and interest rate options

- Volatilities vary in strike (*smile*)
- Volatilities vary in time to maturity (*term structure*)
- Volatility clustering

Fourier and Laplace based valuation formulas

Carr and Madan (1999)

Raible (2000)

Borovkov and Novikov (2002): exotic options

Hubalek, Kallsen, and Krawczyk (2006): hedging

Lee (2004): discretization error in fast Fourier transform

Hubalek and Kallsen (2005): options on several assets

Biagini, Bregman, and Meyer-Brandis (2008): indices

Hurd and Zhou (2009): spread options

Eberlein and Kluge (2006): interest rate derivatives

Eberlein and Koval (2006): cross currency derivatives

Eberlein, Kluge, and Schönbucher (2006): credit default swaptions

Harmonic analysis (Parseval's formula)

Exponential semimartingale model

$\mathcal{B}_T = (\Omega, \mathcal{F}, \mathbf{F}, P)$ stochastic basis, where $\mathcal{F} = \mathcal{F}_T$ and $\mathbf{F} = (\mathcal{F}_t)_{0 \le t \le T}$.
Price process of a financial asset as exponential semimartingale

$$S_t = S_0 \mathrm{e}^{H_t}, \quad 0 \le t \le T. \tag{1}$$

$H = (H_t)_{0 \le t \le T}$ semimartingale with canonical representation

$$H = B + H^c + h(x) * (\mu^H - \nu) + (x - h(x)) * \mu^H. \tag{2}$$

For the processes B, $C = \langle H^c \rangle$, and the measure ν we use the notation

$$\mathbb{T}(H|P) = (B, C, \nu)$$

which is called the *triplet of predictable characteristics* of H.

Alternative model description

$\mathcal{E}(X) = (\mathcal{E}(X)_t)_{0 \le t \le T}$ stochastic exponential

$$\begin{aligned} S_t &= \mathcal{E}(\widetilde{H})_t, \quad 0 \le t \le T \\ dS_t &= S_{t-} d\widetilde{H}_t \end{aligned}$$

where

$$\widetilde{H}_t = H_t + \frac{1}{2}\langle H^c \rangle_t + \int_0^t \int_{\mathbb{R}} (e^x - 1 - x)\mu^H(\mathrm{d}s, \mathrm{d}x)$$

Note

$$\mathcal{E}(\widetilde{H})_t = \exp\left(\widetilde{H}_t - \frac{1}{2}\langle \widetilde{H}^c \rangle_t\right) \prod_{0<s\le t} (1 + \Delta\widetilde{H}_s)\exp(-\Delta\widetilde{H}_s)$$

Asset price positive only if $\Delta\widetilde{H} > -1$.

Martingale modeling

Let $\mathcal{M}_{\text{loc}}(P)$ be the class of local martingales.

Assumption $(\mathbb{ES})$

The process $\mathbb{1}_{\{x>1\}} e^x * \nu$ *has bounded variation.*

Then

$$S = S_0 e^H \in \mathcal{M}_{\text{loc}}(P) \Leftrightarrow B + \frac{C}{2} + (e^x - 1 - h(x)) * \nu = 0. \qquad (3)$$

Throughout, we assume that P is an equivalent martingale measure for S.

By the *Fundamental Theorem of Asset Pricing*, the value of an option on S equals the *discounted expected payoff* under this martingale measure.

We assume *zero* interest rates.

Supremum and infimum processes

Let $X = (X_t)_{0 \leq t \leq T}$ be a stochastic process. Denote by

$$\overline{X}_t = \sup_{0 \leq u \leq t} X_u \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq u \leq t} X_u$$

the supremum and infimum process of X respectively. Since the exponential function is monotone and increasing

$$\overline{S}_T = \sup_{0 \leq t \leq T} S_t = \sup_{0 \leq t \leq T} \left(S_0 e^{H_t} \right) = S_0 e^{\sup_{0 \leq t \leq T} H_t} = S_0 e^{\overline{H}_T}. \qquad (4)$$

Similarly

$$\underline{S}_T = S_0 e^{\underline{H}_T}. \qquad (5)$$

Valuation formulas – payoff functional

We want to price an option with payoff $\Phi(S_t, 0 \leq t \leq T)$, where Φ is a measurable, non-negative functional.

Separation of payoff function from the underlying process:

Example

Fixed strike lookback option

$$(\overline{S}_T - K)^+ = (S_0 \, \mathrm{e}^{\overline{H}_T} - K)^+ = \left(\mathrm{e}^{\overline{H}_T + \log S_0} - K\right)^+$$

1. The *payoff function* is an arbitrary function $f : \mathbb{R} \to \mathbb{R}_+$; for example $f(x) = (\mathrm{e}^x - K)^+$ or $f(x) = \mathbb{1}_{\{\mathrm{e}^x > B\}}$, for $K, B \in \mathbb{R}_+$.
2. The *underlying process* denoted by X, can be the log-asset price process or the supremum/infimum or an average of the log-asset price process (e.g. $X = H$ or $X = \overline{H}$).

Valuation formulas

Consider the option price as a function of S_0 or better of $s = -\log S_0$

X driving process ($X = H, \overline{H}, \underline{H}$, etc.)

$$\Rightarrow \qquad \Phi(S_0 \, \mathrm{e}^{H_t}, 0 \le t \le T) = f(X_T - s)$$

Time-0 price of the option (assuming $r \equiv 0$)

$$\mathbb{V}_f(X; s) = E\big[\Phi(S_t, 0 \le t \le T)\big] = E[f(X_T - s)]$$

Valuation formulas based on Fourier and Laplace transforms

Carr and Madan (1999) plain vanilla options

Raible (2000) general payoffs, Lebesgue densities

In these approaches: Some sort of continuity assumption (payoff or random variable)

Valuation formulas – assumptions

M_{X_T} moment generating function of X_T

$g(x) = \mathrm{e}^{-Rx} f(x)$ (for some $R \in \mathbb{R}$) dampened payoff function

$L^1_{bc}(\mathbb{R})$ bounded, continuous functions in $L^1(\mathbb{R})$

Assumptions

(C1) $g \in L^1_{bc}(\mathbb{R})$

(C2) $M_{X_T}(R)$ *exists*

(C3) $\widehat{g} \in L^1(\mathbb{R})$

Valuation formulas

Theorem

Assume that (C1)–(C3) are in force. Then, the price $\mathbb{V}_f(X;s)$ of an option on $S=(S_t)_{0\leq t\leq T}$ with payoff $f(X_T)$ is given by

$$\mathbb{V}_f(X;s)=\frac{\mathrm{e}^{-Rs}}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{ius}\varphi_{X_T}(-u-iR)\widehat{f}(u+iR)\mathrm{d}u, \tag{6}$$

where φ_{X_T} denotes the extended characteristic function of X_T and $\widehat{f}$ denotes the Fourier transform of f.

Discussion of assumptions

Alternative choice: (C1') $g \in L^1(\mathbb{R})$

(C3') $\widehat{e^{R \cdot} P_{X_T}} \in L^1(\mathbb{R})$

(C3') $\Longrightarrow e^{R \cdot} P_{X_T}$ has a cont. bounded Lebesgue density

Recall: (C3) $\widehat{g} \in L^1(\mathbb{R})$

Sobolov space

$$H^1(\mathbb{R}) = \{g \in L^2(\mathbb{R}) \mid \partial g \text{ exists and } \partial g \in L^2(\mathbb{R})\}$$

Lemma

$g \in H^1(\mathbb{R}) \Longrightarrow \widehat{g} \in L^1(\mathbb{R})$

Similar for the Sobolev–Slobodeckij space $H^S(\mathbb{R})$ $(s > \frac{1}{2})$

Examples of payoff functions

Example (Call and put option)

Call payoff $f(x) = (\mathrm{e}^x - K)^+$, $K \in \mathbb{R}_+$,

$$\widehat{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)}, \qquad R \in I_1 = (1, \infty). \tag{7}$$

Similarly, if $f(x) = (K - \mathrm{e}^x)^+$, $K \in \mathbb{R}_+$,

$$\widehat{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)}, \qquad R \in I_1 = (-\infty, 0). \tag{8}$$

Example (Digital option)

Call payoff $\mathbb{1}_{\{e^x > B\}}$, $B \in \mathbb{R}_+$.

$$\widehat{f}(u+iR) = -B^{iu-R}\frac{1}{iu-R}, \qquad R \in I_1 = (0,\infty). \tag{9}$$

Similarly, for the payoff $f(x) = \mathbb{1}_{\{e^x < B\}}$, $B \in \mathbb{R}_+$,

$$\widehat{f}(u+iR) = B^{iu-R}\frac{1}{iu-R}, \qquad R \in I_1 = (-\infty,0). \tag{10}$$

Example (Double digital option)

The payoff of a double digital option is $\mathbb{1}_{\{\underline{B} < e^x < \overline{B}\}}$, $\underline{B}, \overline{B} \in \mathbb{R}_+$.

$$\widehat{f}(u+iR) = \frac{1}{iu-R}\left(\overline{B}^{iu-R} - \underline{B}^{iu-R}\right), \qquad R \in I_1 = \mathbb{R}\backslash\{0\}. \tag{11}$$

Example (Asset-or-nothing digital)

Payoff $f(x) = \mathrm{e}^x \mathbb{1}_{\{\mathrm{e}^x > B\}}$

$$\widehat{f}(u+iR) = -\frac{B^{1+iu-R}}{1+iu-R}, \quad R \in I_1 = (1, \infty)$$

Similarly $f(x) = \mathrm{e}^x \mathbb{1}_{\{\mathrm{e}^x < B\}}$

$$\widehat{f}(u+iR) = \frac{B^{1+iu-R}}{1+iu-R}, \quad R \in I_1 = (-\infty, 1)$$

Example (Self-quanto option)

Call payoff $f(x) = \mathrm{e}^x (\mathrm{e}^x - K)^+$

$$\widehat{f}(u+iR) = \frac{K^{2+iu-R}}{(1+iu-R)(2+iu-R)}, \quad R \in I_1 = (2, \infty)$$

Non-path-dependent options

European option on an asset with price process $S_t = e^{H_t}$

Examples: call, put, digitals, asset-or-nothing, double digitals, self-quanto options

$\longrightarrow \quad X_T \equiv H_T, \quad$ i.e. we need φ_{H_T}

Generalized hyperbolic model (GH model): Eberlein, Keller (1995), Eberlein, Keller, Prause (1998), Eberlein (2001)

$$\varphi_{H_1}(u) = e^{iu\mu}\Big(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\Big)^{\lambda/2}\frac{K_\lambda\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\,\big)}{K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\big)}$$

$$I_2 = (-\alpha-\beta, \alpha-\beta)$$

$$\varphi_{H_T}(u) = \left(\varphi_{H_1}(u)\right)^T$$

similar: NIG, CGMY, Meixner

Non-path-dependent options II

Stochastic volatility Lévy models: Carr, Geman, Madan, Yor (2003) Eberlein, Kallsen, Kristen (2003)

Stochastic clock $Y_t = \int_0^t y_s ds \quad (y_s > 0)$

e.g. CIR process

$$dy_t = K(\eta - y_t)dt + \lambda y_t^{1/2} dW_t$$

Define for a pure jump Lévy process $X = (X_t)_{t \geq 0}$

$$H_t = X_{Y_t} \quad (0 \leq t \leq T)$$

Then

$$\varphi_{H_t}(u) = \frac{\varphi_{Y_t}(-i\varphi_{X_t}(u))}{(\varphi_{Y_t}(-iu\varphi_{X_t}(-i)))^{iu}}$$

Classification of option types

Lévy model $S_t = S_0 \mathrm{e}^{H_t}$

payoff	payoff function	distributional properties
$(S_T - K)^+$ call	$f(x) = (e^x - K)^+$	P_{H_T} usually has a density
$\mathbb{1}_{\{S_T > B\}}$ digital	$f(x) = \mathbb{1}_{\{e^x > B\}}$	–″–
$(\overline{S}_T - K)^+$ lookback	$f(x) = (e^x - K)^+$	density of $P_{\overline{H}_T}$?
$\mathbb{1}_{\{\overline{S}_T > B\}}$ digital barrier = one touch	$f(x) = \mathbb{1}_{\{e^x > B\}}$	–″–

Valuation formula for the last case

Payoff function f maybe discontinuous
P_{X_T} does not necessarily possess a Lebesgue density

Assumption

(D1) $g \in L^1(\mathbb{R})$

(D2) $M_{X_T}(R)$ *exists*

Theorem

Assume (D1)–(D2) then

$$\mathbb{V}_f(X;s) = \lim_{A\to\infty} \frac{e^{-Rs}}{2\pi} \int_{-A}^{A} \mathrm{e}^{-ius} \varphi_{X_T}(u - iR)\widehat{f}(iR - u)\,\mathrm{d}u \qquad (12)$$

if $\mathbb{V}_f(X;\cdot)$ *is of bounded variation in a neighborhood of* s *and* $\mathbb{V}_f(X;\cdot)$ *is continuous at* s*.*

Options on multiple assets

Basket options

Options on the minimum: $(S_T^1 \wedge \cdots \wedge S_T^d - K)^+$

Multiple functionals of one asset

Barrier options: $(S_T - K)^+ \mathbb{1}_{\{\overline{S}_T > B\}}$

Slide-in or corridor options: $(S_T - K)^+ \sum_{i=1}^{N} \mathbb{1}_{\{L < S_{T_i} < H\}}$

Modelling:
$$S_t^i = S_0^i \exp(H_t^i) \quad (1 \leq i \leq d)$$
$$X_T = \Psi(H_t \mid 0 \leq t \leq T)$$
$$f : \mathbb{R}^d \longrightarrow \mathbb{R}_+$$
$$g(x) = e^{-\langle R, x \rangle} f(x) \quad (x \in \mathbb{R}^d)$$

Assumptions:

(A1) $g \in L^1(\mathbb{R}^d)$

(A2) $M_{X_T}(R)$ exists

(A3) $\widehat{\varrho} \in L^1(\mathbb{R}^d)$ where $\varrho(dx) = e^{\langle R, x \rangle} P_{X_T}(dx)$

Options on multiple assets (cont.)

Theorem

If the asset price processes are modeled as exponential semimartingale processes such that $S^i \in \mathcal{M}_{\mathrm{loc}}(P)$ $(1 \leq i \leq d)$ *and conditions (A1)–(A3) are in force, then*

$$\mathbb{V}_f(X; s) = \frac{e^{-\langle R, s \rangle}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u, s \rangle} M_{X_T}(R + iu) \widehat{f}(iR - u) du$$

Remark

When the payoff function is discontinuous and the driving process does not possess a Lebesgue density $\longrightarrow$ L^2-limit result

Sensitivities – Greeks

$$\mathbb{V}_f(X;S_0) = \frac{1}{2\pi}\int_{\mathbb{R}} S_0^{R-\mathrm{i}u} M_{X_T}(R-\mathrm{i}u)\widehat{f}(u+\mathrm{i}R)du$$

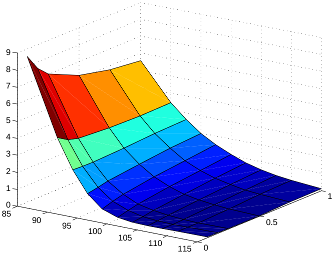
Delta of an option

$$\Delta_f(X;S_0) = \frac{\partial \mathbb{V}(X;S_0)}{\partial S_0} = \frac{1}{2\pi}\int_{\mathbb{R}} S_0^{R-1-\mathrm{i}u} M_{X_T}(R-\mathrm{i}u)\frac{\widehat{f}(u+\mathrm{i}R)}{(R-\mathrm{i}u)^{-1}}du$$

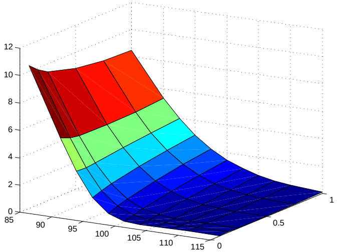
Gamma of an option

$$\Gamma_f(X;S_0) = \frac{\partial^2 \mathbb{V}_f(X;S_0)}{\partial^2 S_0} = \frac{1}{2\pi}\int_{\mathbb{R}} S_0^{R-2-\mathrm{i}u} \frac{M_{X_T}(R-\mathrm{i}u)\widehat{f}(u+\mathrm{i}R)}{(R-1-\mathrm{i}u)^{-1}(R-\mathrm{i}u)^{-1}}du$$

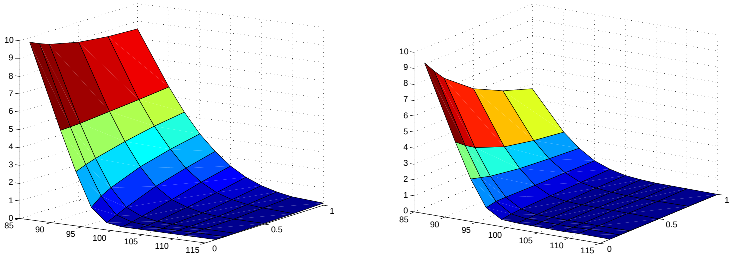
Numerical examples



Option prices in the 2d Black-Scholes model with negative correlation.



Option prices in the 2d stochastic volatility model.



Option prices in the 2d GH model with positive (left) and negative (right) correlation.

Lévy processes

Let $L = (L_t)_{0 \leq t \leq T}$ be a *Lévy process* with triplet of local characteristics (b, c, λ), i.e. $B_t(\omega) = bt$, $C_t(\omega) = ct$, $\nu(\omega; dt, dx) = dt\lambda(\mathrm{d}x)$, λ Lévy measure.

Assumption $(\mathbb{EM})$

There exists a constant $M > 1$ such that

$$\int_{\{|x|>1\}} \mathrm{e}^{ux}\lambda(\mathrm{d}x) < \infty, \qquad \forall u \in [-M, M].$$

Using $(\mathbb{EM})$ and Theorems 25.3 and 25.17 in Sato (1999), we get that

$$E\left[\mathrm{e}^{uL_t}\right] < \infty, \quad E\left[\mathrm{e}^{u\overline{L}_t}\right] < \infty \quad \text{and} \quad E\left[\mathrm{e}^{u\underline{L}_t}\right] < \infty$$

for all $u \in [-M, M]$.

On the characteristic function of the supremum I

Proposition

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process that satisfies assumption $(\mathbb{EM})$. Then, the characteristic function $\varphi_{\overline{L}_t}$ of $\overline{L}_t$ has an analytic extension to the half plane $\{z \in \mathbb{C} : -M < \Im z < \infty\}$ and can be represented as a Fourier integral in the complex domain

$$\varphi_{\overline{L}_t}(z) = E\left[\mathrm{e}^{iz\overline{L}_t}\right] = \int_{\mathbb{R}} \mathrm{e}^{izx} P_{\overline{L}_t}(\mathrm{d}x).$$

Fluctuation theory for Lévy processes

Theorem
(Extension of Wiener–Hopf to the complex plane)

Let L be a Lévy process. The Laplace transform of $\overline{L}$ at an independent and exponentially distributed time θ, $\theta \sim \mathrm{Exp}(q)$, can be identified from the *Wiener–Hopf factorization* of L via

$$E\left[\mathrm{e}^{-\beta \overline{L}_\theta}\right] = \int_0^\infty q E[e^{-\beta \overline{L}_t}] e^{-qt}\, \mathrm{d}t = \frac{\kappa(q,0)}{\kappa(q,\beta)} \tag{13}$$

for $q > \alpha^*(M)$ and $\beta \in \{\beta \in \mathbb{C} | \mathcal{R}(\beta) > -M\}$ where $\kappa(q,\beta)$, is given by

$$\kappa(q,\beta) = k \exp\left(\int_0^\infty \int_0^\infty (e^{-t} - e^{-qt-\beta x}) \frac{1}{t} P_{L_t}(\mathrm{d}x)\, \mathrm{d}t \right). \tag{14}$$

On the characteristic function of the supremum II

Theorem

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process satisfying assumption $(\mathbb{EM})$. The Laplace transform of $\overline{L}_t$ at a fixed time t, $t \in [0, T]$, is given by

$$E\left[e^{-\beta \overline{L}_t}\right] = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{t(Y+iv)}}{Y + iv} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, \beta)} dv, \tag{15}$$

for $Y > \alpha^*(M)$ and $\beta \in \mathbb{C}$ with $\Re\beta \in (-M, \infty)$.

Remark

Note that $\beta = -iz$ provides the characteristic function.

Application to lookback options

Fixed strike lookback call: $(\overline{S}_T - K)^+$ (analogous for lookback put).

Combining the results, we get

$$\mathbb{C}_T(\overline{S}; K) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-iu} \varphi_{\overline{L}_T}(-u - iR) \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)} \mathrm{d}u \qquad (16)$$

where

$$\varphi_{\overline{L}_T}(-u - iR) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{\mathrm{e}^{T(Y+iv)}}{Y + iv} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, iu - R)} \mathrm{d}v \qquad (17)$$

for $R \in (1, M)$ and $Y > \alpha^*(M)$.

- The floating strike lookback option, $(\overline{S}_T - S_T)^+$, is treated by a *duality* formula (Eb., Papapantoleon (2005)).

One-touch options

One-touch call option: $\mathbb{1}_{\{\overline{S}_T > B\}}$.

Driving Lévy process L is assumed to have infinite variation or has infinite activity and is regular upwards. L satisfies assumption $(\mathbb{EM})$, then

$$\begin{aligned} \mathbb{DC}_T(\overline{S}; B) &= \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} S_0^{R+iu} \varphi_{\overline{L}_T}(u - iR) \frac{B^{-R-iu}}{R + iu} \mathrm{d}u \\ &= P(\overline{L}_T > \log(B/S_0)) \end{aligned} \tag{18}$$

for $R \in (0, M)$.

Equity default swap (EDS)

- Fixed premium exchanged for payment at “default”
- default: drop of stock price by 30 % or 50 % of S_0 $\rightarrow$ first passage time
- fixed leg pays premium $\mathcal{K}$ at times $T_1, \dots, T_N$, if $T_i \leq \tau_B$
- if $\tau_B \leq T$: protection payment C, paid at time τ_B
- premium of the EDS chosen such that initial value equals 0; hence

$$\mathcal{K} = \frac{C E\left[\mathrm{e}^{-r\tau_B} \mathbb{1}_{\{\tau_B \leq T\}}\right]}{\sum_{i=1}^{N} E\left[\mathrm{e}^{-rT_i} \mathbb{1}_{\{\tau_B > T_i\}}\right]}. \tag{19}$$

- Calculations similar to touch options, since $\mathbb{1}_{\{\tau_B \leq T\}} = \mathbb{1}_{\{\underline{S}_T \leq B\}}$.

Basic interest rates

$B(t,T)$: price at time $t \in [0,T]$ of a default-free zero coupon bond with maturity $T \in [0,T^*]$ $\quad (B(T,T)=1)$

$f(t,T)$: instantaneous forward rate

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

$L(t,T)$: default-free forward Libor rate for the interval T to $T+\delta$ as of time $t \le T$ $\quad$ (δ-forward Libor rate)

$$L(t,T) := \tfrac{1}{\delta}\left(\tfrac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

$F_B(t,T,U)$: forward price process for the two maturities $T < U$

$$F_B(t,T,U) := \tfrac{B(t,T)}{B(t,U)}$$

$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$

Dynamics of the forward rates

(Eb–Raible (1999), Eb–Özkan (2003), Eb–Jacod–Raible (2005), Eb–Kluge (2006)

$$\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t - \sigma(t,T)\,\mathrm{d}L_t \qquad (0 \le t \le T \le T^*)$$

$\alpha(t,T)$ and $\sigma(t,T)$ satisfy measurability and boundedness conditions and $\alpha(s,T) = \sigma(s,T) = 0$ for $s > T$

Define $\quad A(s,T) = \int_{s\wedge T}^{T} \alpha(s,u)\,\mathrm{d}u \quad$ and $\quad \Sigma(s,T) = \int_{s\wedge T}^{T} \sigma(s,u)\,\mathrm{d}u$

Assume $0 \le \Sigma^i(s,T) \le M \qquad (1 \le i \le d)$

For most purposes we can consider deterministic α and σ

Implications

Savings account and default-free zero coupon bond prices are given by

$$B_t = \frac{1}{B(0,t)} \exp\left(\int_0^t A(s,T)\,\mathrm{d}s - \int_0^t \Sigma(s,t)\,\mathrm{d}L_s \right) \quad \text{and}$$

$$B(t,T) = B(0,T)B_t \exp\left(-\int_0^t A(s,T)\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s \right).$$

If we choose $A(s,T) = \theta_s(\Sigma(s,T))$, then bond prices, discounted by the savings account, are martingales.

In case $d = 1$, the martingale measure is unique (see Eberlein, Jacod, and Raible (2004)).

Key tool

$L = (L^1, \dots, L^d)$ d-dimensional time-inhomogeneous Lévy process

$$\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(iu)\,\mathrm{d}s \qquad \text{where}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x)$$

in case L is a (time-homogeneous) Lévy process, $\theta_s = \theta$ is the cumulant (log-moment generating function) of L_1.

Proposition Eberlein, Raible (1999)

Suppose $f : \mathbb{R}_+ \to \mathbb{C}^d$ is a continuous function such that $|\mathcal{R}(f^i(x))| \leq M$ for all $i \in \{1, \dots, d\}$ and $x \in \mathbb{R}_+$, then

$$\mathbb{E}\left[\exp\left(\int_0^t f(s)dL_s\right)\right] = \exp\left(\int_0^t \theta_s(f(s))ds\right)$$

Take $f(s) = \sum(s, T)$ for some $T \in [0, T^*]$

Pricing of European options

$$B(t,T) = B(0,T)\exp\left[\int_0^t (r(s) + \theta_s(\Sigma(s,T)))\,\mathrm{d}s + \int_0^t \Sigma(s,T)\mathrm{d}L_s\right]$$

where $r(t) = f(t,t)$ short rate

$V(0,t,T,w)$ time-0-price of a European option with maturity t and payoff $w(B(t,T),K)$

$$V(0,t,T,w) = \mathbb{E}_{\mathbb{P}^*}[B_t^{-1} w(B(t,T),K)]$$

Volatility structures

$$\Sigma(t,T) = \frac{\widehat{\sigma}}{a}(1 - \exp(-a(T-t))) \qquad \text{(Vasiček)}$$

$$\Sigma(t,T) = \widehat{\sigma}(T-t) \qquad \text{(Ho–Lee)}$$

Fast algorithms for Caps, Floors, Swaptions, Digitals, Range options

Pricing formula for caps

(Eberlein, Kluge (2006))

$$w(B(t,T),K) = (B(t,T)-K)^+$$

Call with strike K and maturity t on a bond that matures at T

$$\begin{aligned} C(0,t,T,K) &= \mathbb{E}_{\mathbb{P}^*}[B_t^{-1}(B(t,T)-K)^+] \\ &= B(0,t)\mathbb{E}_{\mathbb{P}_t}[(B(t,T)-K)^+] \end{aligned}$$

Assume $X = \int_0^t (\Sigma(s,T)-\Sigma(s,t))dL_s$ has a Lebesgue density, then

$$\begin{aligned} C(0,t,T,K) &= \frac{1}{2\pi}KB(0,t)\exp(R\xi) \\ &\quad \times \int_{-\infty}^{\infty} e^{iu\xi}(R+iu)^{-1}(R+1+iu)^{-1}M_t^X(-R-iu)du \end{aligned}$$

where ξ is a constant and $R < -1$.

Analogous for the corresponding put and for swaptions

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