American Option Pricing in a Markov Chain Market Model

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Abstract

This work represents joint research with John van der Hoek from University of South Australia, School of Mathematics and Statistics.

This paper is a sequel to our previous paper "A New Paradigm in Asset Pricing" in which we construct a model for asset pricing in a world where the randomness is modeled by a Markov chain. In this paper we develop a theory of optimal stopping and related variational inequalities for American options in this model. A version of Saigal's Lemma is established and numerical results obtained.

Overview

1. Introduction and Stock Price Processes.
2. European Options.
3. American Claims.
4. Existence and Uniqueness.
5. Numerical Solution of the Variational Inequality.

1. Stochastic Discounting Function Processes

We will assume that we have a probability space $(\Omega, \mathcal{F}, P)$ on which there is defined a stochastic discounting function process $\{\pi_t \mid t \geq 0\}$ with the property that for any asset price process $\{\mathcal{A}_t | t \geq 0\}$

$$\pi_t \mathcal{A}_t = \mathsf{E}[\pi_s \mathcal{A}_s | \mathcal{F}_t]$$

where E is expectation with respect to P and $\mathcal{F}_t$ represents information up to time t. Here $s \geq t$ and there are no cash-flows from $\mathcal{A}$ over the time interval $(t, s]$.

We will abbreviate Stochastic Discounting Function to SDF.

Structure of the SDF

We consider a simple dynamic model of an economic world where, rather than using Brownian motions and related diffusions, the uncertainty is modelled by a finite state Markov chain $X = \{X_t, t \geq 0\}$.

The chain is defined on a probability space $(\Omega, \mathcal{F}, P)$. Here initially P will denote the historical probability.

As in Elliott, Aggoun and Moore (1995/2008), the finite state space of the chain X can be identified with the set of unit vectors $\{e_1, e_2, \ldots, e_N\}$, where

$$e_i = (0, \ldots, 0, 1, \ldots, 0)' \in R^N.$$

The semimartingale representation of the chain X is then

$$X_t = X_0 + \int_0^t A_u X_u du + M_t \in R^N.$$

Here for each $u \geq 0$, A_u is an irreducible rate matrix.

As in van der Hoek and Elliott (2008a) we suppose the SDF process $\pi = \{\pi_t, t \geq 0\}$ has the form

$$\pi_t = \exp\Big[-\int_0^t X'_{u-} C_{u-} dX_u - \int_0^t D'_u X_u du\Big].$$

Here, for each $u \geq 0$, C_u is an $N \times N$ matrix and D_u a vector in R^N.

Of necessity $\pi = \{\pi_t, t \geq 0\}$ should be a potential process. This places some additional restrictions on C and D.

Necessary Properties

The process $\{\pi_t\}$ must satisfy three conditions:

1. be strictly positive, to prevent arbitrage opportunities.

2. be a super-martingale: this means for $s \geq t$ that

$$\mathsf{E}\left[\pi_s | \mathcal{F}_t\right] \leq \pi_t \ .$$

This is because

$$\mathsf{E}\left[\pi_s | \mathcal{F}_t\right] = \pi_t P(t,s)$$

and $P(t,s)$ is the value at t of one dollar at time s.

3. for any $t \geq 0$,

$$\mathsf{E}\left[\pi_s | \mathcal{F}_t\right] \to 0$$

as $s \to \infty$, as a dollar at infinity is worth zero now.

In probability theory such a process is called a **potential**.

Write

$$\Lambda_{t,T} = \frac{\pi_T}{\pi_t} \quad \text{and} \quad Z_{t,T} = \Lambda_{t,T} \cdot X_T$$

Then it can be shown that

$$\begin{aligned}\widehat{Z}_{t,T} &= E[\Lambda_{t,T} X_T | \mathcal{F}_t] \\ &= E[\Lambda_{t,T} X_T | X_t] \\ &= \Psi(t,T) X_t\end{aligned}$$

where Ψ is a matrix function which is the solution of

$$\begin{gathered}\frac{\partial \Psi(t,T)}{\partial T} = \Gamma_T \Psi(t,T) \\ \Psi(t,t) = I_N.\end{gathered}$$

Here Γ_u is an $N \times N$ matrix with

$$\begin{aligned} \Gamma_u^{ij} &= A_u^{ij} \exp(C_u^{jj} - C_u^{ji}) \qquad &\text{if} \qquad &i \neq j \\ \Gamma_u^{jj} &= A_u^{jj} - D_u^j \qquad &\text{if} \qquad &i = j. \end{aligned}$$

In this model a zero coupon bond has the price

$$\begin{aligned} P(t,T) &= E[\Lambda_{t,T} | \mathcal{F}_t] \\ &= E[\langle Z_{t,T}, 1 \rangle | \mathcal{F}_t] \\ &= \langle \widehat{Z}_{t,T}, 1 \rangle \\ &= \langle \Psi(t,T) X_t, 1 \rangle \,. \end{aligned}$$

Now $\widehat{Z}_{t,T} = X_t + \int_t^\infty \Gamma_u \Psi_{t,u} X_t du$ so $P(t,T) = 1 + \int_t^T \langle \Gamma_u \Psi_{t,u} X_t, 1 \rangle du$. This price should be ≤ 1 which implies (componentwise) that

$$\Gamma_u^* 1 \equiv \Gamma_u' 1 \leq 0.$$

This is the case if the components of D_u are large enough.

Stock Price Processes

Suppose a stock has a random dividend rate process $\mathcal{D} = \{\mathcal{D}_s,\ s \geq 0\}$. Suppose

$$\mathcal{D}_s = \langle \delta_s, X_s \rangle$$

where $\delta_s \in R^N$, so the dividend rate depends on the state of the world.

Using the SDF the present value of future dividend payments gives the stock price as

$$\begin{aligned} S_t &= \frac{1}{\pi_t} E\Big[\int_t^\infty \pi_s \mathcal{D}_s ds | \mathcal{F}_t \Big] \\ &= \frac{1}{\pi_t} E\Big[\int_t^\infty \pi_s \langle \delta_s, X_s \rangle ds | X_t \Big] \\ &= \langle \sigma_t, X_t \rangle, \quad \text{say.} \end{aligned}$$

Note that $\pi_t S_t + \int_0^t \pi_s \mathcal{D}_s ds$ is a martingale. However,

$$\begin{aligned} S_t &= \int_t^\infty E[\Lambda_{t,s}\langle \delta_s, X_s\rangle | X_t] ds \\ &= \int_t^\infty \langle \delta_s, \widehat{Z}_{t,s}\rangle ds \\ &= \int_t^\infty \langle \delta_s, \Psi(t,s) X_t\rangle ds \\ &= \int_t^\infty \langle \Psi(t,s)^* \delta_s, X_t\rangle ds \\ &= \left\langle \int_t^\infty \Psi(t,s)^* \delta_s ds, X_t \right\rangle ds \end{aligned}$$

so

$$\sigma_t = \int_t^\infty \Psi(t,s)^* \delta_s ds.$$

σ satisfies the system of ODEs:

$$\frac{d\sigma_t}{dt} = -\Gamma_t^* \sigma_t - \delta_t$$

with $\sigma_t \to 0$ as $t \to \infty$.

2. European Options

Let us first consider a European claim with payoff at expiry T, of the form $C_T = G(S_T)$. We wish to determine the value of this at time $t \le T$. Write

$$g_T^i = G(\langle \sigma_T, e_i \rangle), \quad 1 \le i \le N,$$

$$g_T = (g_T^1, \dots, g_T^N)' \in R^N.$$

Then $C_T = \langle g_T, X_T \rangle$ so

$$\begin{aligned} C_t &= E[\Lambda_{t,T} C_T | \mathcal{F}_t] \\ &= \langle g_T, \widehat{Z}_{t,T} \rangle \\ &= \langle g_T, \Psi(t,T) X_t \rangle \\ &= \langle \Psi(t,T)^* g_T, X_t \rangle \\ &= \langle c_t, X_t \rangle \end{aligned}$$

where $c_t = \Psi(t,T)^* g_T$.

Thus c is then the solution of a Black-Scholes-Morton type equation

$$\frac{dc_t}{dt} + \Gamma_t^* c_t = 0$$

with

$$C_T = g_T.$$

Remark:

It is open question whether $C_t = u(t, S_t)$ for some function u holds for any dividend structure defining S.

3. American Claims

Consider an American type claim on the asset $S = \{S_t, t \geq 0\}$ where $t \in [0, T]$, and if the claim is exercised at time t the payoff is

$$G_t = G(t, S_t)$$

Recall $S_t = \langle \sigma_t, X_t \rangle$. Write $g_t^i = G(t, \langle \sigma_t, e_i \rangle)$, $1 \leq i \leq N$ and $g_t = (g_t^1, \ldots, g_t^N)' \in R^N$. Then

$$G_t = G(t, S_t) = \langle g_t, X_t \rangle.$$

The value of the American option at time t is determined by the optimal time τ at which to exercise the option where τ is a stopping time with values in $[t, T]$.

That is, we wish to determine

$$\sup_{t \leq \tau \leq T} \frac{1}{\pi_t} E[\pi_\tau G_\tau | \mathcal{F}_t],$$

or equivalently, $\sup_{t \leq \tau \leq T} E[\Lambda_{t,\tau} \langle g_\tau, X_\tau \rangle | X_t]$ where the supremum is taken over all stopping times with values in $[t, T]$.

Theorem 3.1

Suppose there is a solution $a = \{a_t, \, t \geq 0\}$ of the following linear complementarity problem: $a_t \in R^N$ is a deterministic function of $t \in [0, T]$ such that:

3.1 i) $\dfrac{da_t}{dt} + \Gamma_t^* a_t \leq 0$ (componentwise),

3.1 ii) $a_t \geq g_t$, (that is, $a_t^i \geq g_t^i, \quad i = 1, \dots, N$),

3.1 iii) $a_T = g_T$ and

3.1 iv) $\Big(- \dfrac{da_t^i}{dt} - (\Gamma_t^* a_t)^i \Big)(a_t^i - g_t^i) = 0, \; 1 \leq i \leq N$.

Then $\langle a_t, X_t \rangle = \sup_{t \leq \tau \leq T} E\Big[\dfrac{\pi_\tau}{\pi_t} \langle g_\tau, X_\tau \rangle | \mathcal{F}_t \Big]$, where the supremum is over all stopping times with values in $[t, T]$.

Remark:

The basic lemma that is used to establish this is:

Lemma 3.2

Suppose $a = \{a_t,\ 0 \leq t \leq T\}$ is a solution of the linear complementarity problem (i), (ii), (iii), (iv) of Theorem 3.1. Write

$$\mathcal{A}_t = \langle a_t, X_t \rangle.$$

Then $\{\pi_t \mathcal{A}_t, .\, t \geq 0\}$ is a supermartingale.

We have an alternative formulation with variational inequalities.

Write

$$\begin{aligned}\mathcal{K}(t) =&\{x \in R^N | x^i \geq g_t^i \quad \text{for} \quad 1 \leq i \leq N\} \\ \mathcal{K} =&\{v : [0, \infty) \to R^N \quad \text{which are once} \\ &\qquad \text{continuously differentiable with} \\ &\qquad v_t \in \mathcal{K}(t) \quad \text{for all} \quad t \in [0, T]\}.\end{aligned}$$

Note that if $a = \{a_t, 0 \leq t \leq T\}$ is a solution of the linear complementarity problem 3.1(i), 3.1(ii), 3.1(iii), 3.1(iv) then $a \in \mathcal{K}$.

Definition 3.3

A deterministic function

$$a = \{a_t \in R^N, 0 \le t \le T\}$$

is a solution of the (related) variational inequality if:

i) $a \in \mathcal{K}$

ii) $-\left\langle \frac{da_t}{dt}, v_t - a_t \right\rangle - \langle \Gamma_t^* a_t, v_t - a_t \rangle \ge 0 \qquad \forall v \in \mathcal{K}$

iii) $a_T = g_T \in R^N$.

Theorem 3.4

The function a is a solution of the linear complementarity problem of Theorem 3.1 if and only if a is a solution of the variational inequality of Definition 3.3.

Remark:

It seems easier to relate the linear complementarity problem to the optimal stopping problem, but existence, uniqueness, continuous dependency results are easier to derive via the variational inequality formulation.

We now cite a list of results.

4. Existence and Uniqueness

There is a **unique solution** to the variational inequality.

Any solution of the variational equality **depends continuously** on the input data (Γ, g). From now on we will assume that $u \to (\Gamma_u, g_u)$ has sufficient smoothness.

To establish **existence** we use use the penalty approach.

That is, for $\varepsilon > 0$ we consider the system of ordinary differential equations:

$$- \frac{da_t^\varepsilon}{dt} - \Gamma_t^* a_t^\varepsilon = \frac{1}{\varepsilon}(g_t - a_t^\varepsilon)^+,$$
$$a_T^\varepsilon = g_T^\varepsilon.$$

For each $\varepsilon > 0$, there is a unique solution.

The family of solutions $\{a^\varepsilon \mid \varepsilon > 0\}$ is equicontinuous. This is established by adapting arguments from Bensoussan and Lions (1982).

The Ascoli-Arzela theorem give the convergence of a subsequence of these solutions.

It is then shown that this limit is a solution of the variational inequality. Along the way the boundedness for the first derivative of Γ and the second derivative of g is assumed.

5. Numerical Solution of the Variational Inequality

We discuss the variational inequality:

i) $a \in \mathcal{K}$,

ii) $-\left\langle \frac{da_t}{dt}, v_t - a_t \right\rangle - \langle \Gamma_t^* a_t, v_t - a_t \rangle \geq 0 \qquad \forall v \in \mathcal{K}$,

iii) $a_T = g_T \in R^N$.

Write $\Delta t = T/N$

$$\begin{aligned} a^n &= a_{n\Delta t} \quad n = 0, 1, \ldots, N-1 \\ a^N &= g_T. \end{aligned}$$

The values of a^n will be defined by backward recursion starting with $a^N = g_T$.

Suppose a^{n+1} is determined. a^n will be determined by requiring

$$-\left\langle \frac{a^{n+1}-a^n}{\Delta t}, v-a^n \right\rangle - \langle \Gamma^*_{n\Delta t} a^n, v-a^n\rangle \geq 0$$

and

$$a^n \geq g_{n\Delta t}, \quad \text{for all} \quad v \geq g_{n\Delta t}.$$

That is,

$$-\langle \Gamma^*_{n\Delta t} a^n, v-a^n\rangle + \frac{a}{\Delta t}\langle a^n, v-a^n\rangle \geq \frac{1}{\Delta t}\langle a^{n+1}, v-a^n\rangle .$$

We can show this variational inequality has a solution a^n for Δt sufficiently small.

This problem is of the form:

Find $x \in R^N, \quad x \geq g$ (componentwise) with

$$\langle Ax,\, y - x \rangle \geq \langle b,\, y - x \rangle$$

for all $y \geq g$ (componentwise).

In our case

$$A = \Gamma^*_{n\Delta t} - \frac{1}{\Delta t} I_N,$$

$$b = \frac{a^{n+1}}{\Delta t}, \quad \text{and} \quad x, y, b, g \in R^N.$$

This problem has a unique solution (see Kinderlehrer and Stampacchia, for example).

The problem can be formulated as a linear complementarity problem, (LCP). Write

$$
\begin{aligned}
M &= \frac{1}{\Delta t} I_N - \Gamma^*_{n\Delta t} \\
q &= -\frac{a^{n+1}}{\Delta t} - M g_{n\Delta t} \\
x &= a^n - g_{n\Delta t}.
\end{aligned}
$$

Assume there is a solution x of this linear complementarity problem

$$
\begin{aligned}
w = Mx + q &\geq 0 \\
x &\geq 0 \\
\langle w, x \rangle &= 0.
\end{aligned}
$$

then $a^n = x + g_{n\Delta t}$.

There is an extensive literature on such problems. C.W. Cryer (1983) provides an extensive discussion.

The matrix M has positive diagonal elements and $M_{ij} \leq 0$ if $i \neq j$. These conditions allow the so called Saigal algorithm to provide a solution in N steps for just this situation (see van der Hoek (1998) where a direct proof was provided for the case of triangular M which easily generalizes to this situation as well).

This is the **Saigal algorithm**:

Step 1: Write $I = \{i : q_i < 0\}$

Step 2: If $I = \phi$, Stop, and set $x = 0, \quad w = q$

Step 3: Choose $i \in I$.

Step 3.1: If $M_{ii} \leq 0$ Stop. No solution exists.

Step 3.2: Otherwise, pivot on M_{ii}.

Rename the transformed system $w = Mx + q$.

Drop column i from M and go to Step 1.

Remark:

As M has only N columns this procedure stops after at most N steps. In our case $M_{ii} > 0$ so Step 3.1 does not occur.

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