Calibrating affine stochastic volatility models with jumps
An asymptotic approach

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Joint work with A. Mijatović

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A short (non) fictitious story

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Boss: 'Calibrate model $H(a)$ to market data.'

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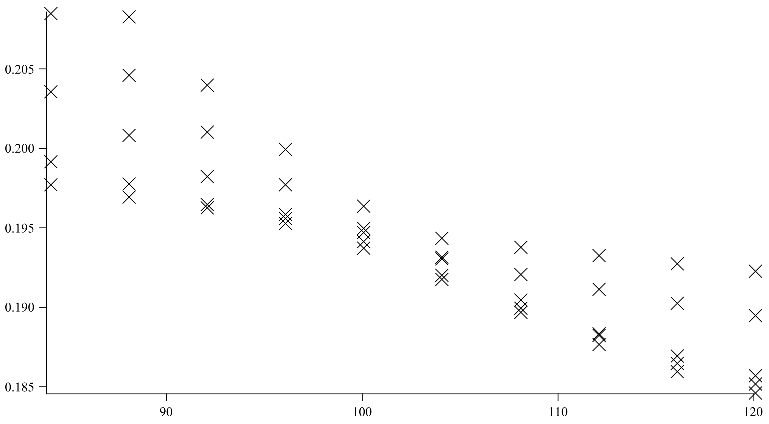

Figure: Market implied volatilities for different strikes and maturities.

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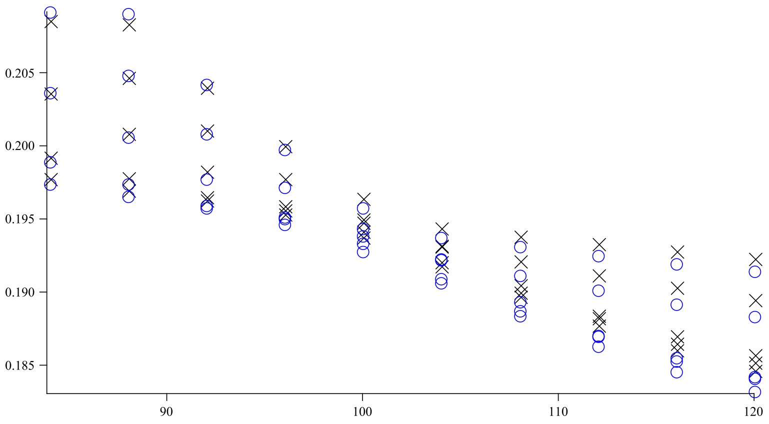

Figure: Sum of squared errors: 4.53061E-05

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Boss: 'Classic mistake!! You should take a_2 instead.'

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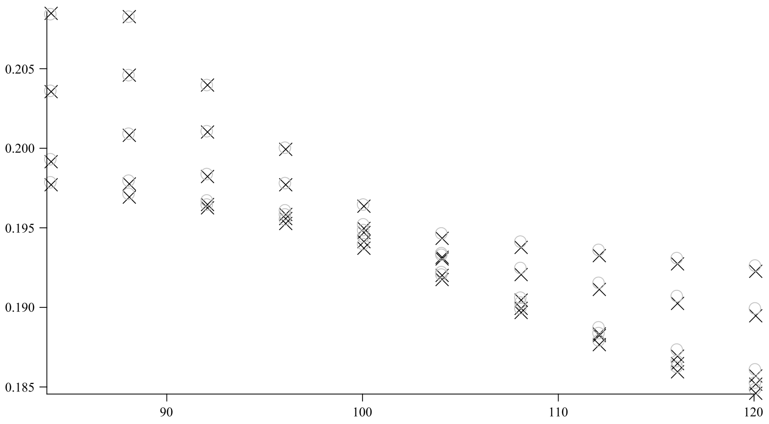

Figure: Sum of squared errors: 2.4856E-06

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Moral of the story:

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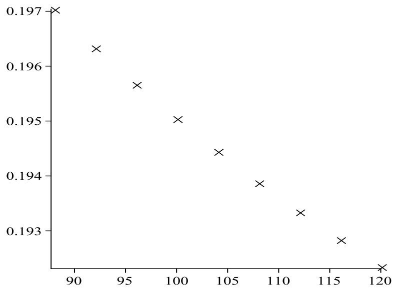
"Start every day off with a smile and get it over with." (W.C. Fields)

So let us start off with a smile (one maturity slice)



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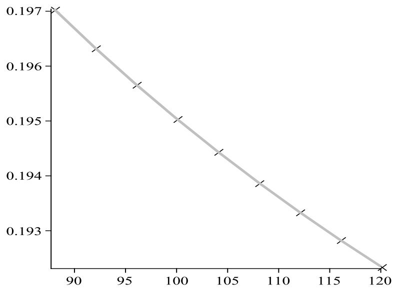

Solid blue: $x \mapsto g(x) := C_{\mathrm{BS}}^{-1}\left(\mathcal{F}^{-1}\Re\left\{f(x,z)\,\phi_a(z)\right\}\right)$

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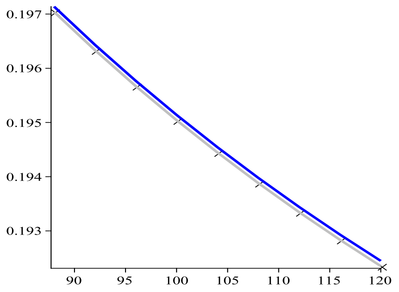

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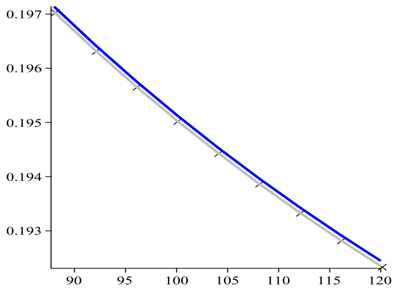

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Easier to calibrate $\hat{g}$ than g.

Motivation and goals

- Obtain closed-form formulae for the implied volatility under ASVM in the short/large-maturity limits.
- Propose an accurate starting point for calibration purposes.
- Discuss conditions on jumps for a model to be usable in practice.

Definition: The implied volatility is the unique parameter $\sigma \geq 0$ such that

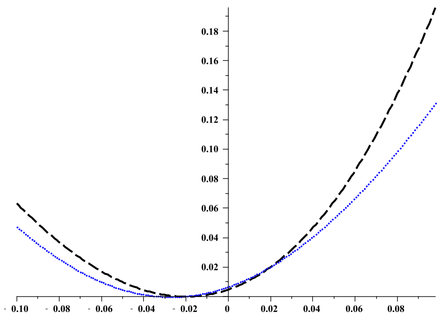
$$C_{\mathrm{BS}}\left(S_0, K, T, \sigma\right) = C_{\mathrm{obs}}\left(S_0, K, T\right).$$

Large deviations theory

Lemma

$(X_\epsilon)_{\epsilon>0}$ satisfies the LDP with the continuous good rate function I if and only if

$$-\lim_{\epsilon\longrightarrow 0}\epsilon\log\mathbb{P}(X_\epsilon\in B)=\inf_{x\in B}I(x),\quad \text{for any set } B\subset\Omega.$$



The Gärtner-Ellis theorem

Assumption A.1: For all $\lambda \in \mathbb{R}$, define the limiting cumulant generating function

$$\Lambda(\lambda) := \lim_{t \to \infty} t^{-1} \log \mathbb{E}\left(\mathrm{e}^{\lambda t X_t}\right) = \lim_{t \to \infty} t^{-1} \Lambda_t(\lambda t)$$

as an extended real number. Denote $\mathcal{D}_\Lambda := \{\lambda \in \mathbb{R} : \Lambda(\lambda) < \infty\}$. Assume further that the origin belongs to $\mathcal{D}_\Lambda^0$.

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Theorem (Gärtner-Ellis) (*special case of the general th. Dembo & Zeitouni*)

Under Assumption A.1, the family of random variables $(X_t)_{t \geq 0}$ satisfies the LDP with rate function Λ^*, defined as the Fenchel-Legendre transform of Λ,

$$\Lambda^*(x) := \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda(\lambda)\}, \quad \text{for all } x \in \mathbb{R}.$$

Methodology overview (large-time)

- Let $(S_t)_{t\geq 0}$ be a share price process, and define $X_t := \log(S_t/S_0)$.

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- Translate the tail behaviour of X into an asymptotic behaviour of Call prices.
- Translate these Call price asymptotics into implied volatility asymptotics.

Affine stochastic volatility models

Let $(S_t)_{t\geq 0}$ represent a share price process and a martingale. Define $X_t := \log S_t$ and assume that $(X_t, V_t)_{t\geq 0}$ is a stochastically continuous, time-homogeneous Markov process satisfying

$$\Phi_t(u,w) := \log \mathbb{E}\left(\mathrm{e}^{uX_t + wV_t} \middle| X_0, V_0\right) = \phi(t,u,w) + V_0 \psi(t,u,w) + uX_0,$$

for all $t, u, w \in \mathbb{R}_+ \times \mathbb{C}^2$ such that the expectation exists.
Define $F(u,w) := \partial_t \phi(t,u,w)|_{t=0^+}$, and $R(u,w) := \partial_t \psi(t,u,w)|_{t=0^+}$. Then

$$F(u,w) = \left\langle \frac{a}{2}\begin{pmatrix} u \\ w \end{pmatrix} + b, \begin{pmatrix} u \\ w \end{pmatrix}\right\rangle + \int_{D\setminus\{0\}} \left(\mathrm{e}^{xu+yw} - 1 - \left\langle \omega_F(x,y), \begin{pmatrix} u \\ w \end{pmatrix}\right\rangle\right) \mathrm{m}(\mathrm{d}x, \mathrm{d}y),$$

$$R(u,w) = \left\langle \frac{\alpha}{2}\begin{pmatrix} u \\ w \end{pmatrix} + \beta, \begin{pmatrix} u \\ w \end{pmatrix}\right\rangle + \int_{D\setminus\{0\}} \left(\mathrm{e}^{xu+yw} - 1 - \left\langle \omega_R(x,y), \begin{pmatrix} u \\ w \end{pmatrix}\right\rangle\right) \mu(\mathrm{d}x, \mathrm{d}y),$$

where $D := \mathbb{R} \times \mathbb{R}_+$, and ω_F and ω_R are truncation functions.
See Duffie, Filipović, Schachermayer (2003) and Keller-Ressel (2009).

Why this class of models?

- They feature most market characteristics: jumps, stochastic volatility, ...
- Their analytic properties are known (Duffie, Filipović & Schachermayer).
- They are tractable and pricing can be performed using Carr-Madan or Lewis inverse Fourier transform method.
- Most models used in practice fall into this category: Heston, Bates, exponential Lévy models (VG, CGMY), pure jump process (Merton, Kou), Barndorff-Nielsen & Shephard, ...

Large-time asymptotics: objectives and tools

Recall that $\Lambda_t(u,w) := \phi(t,u,w) + V_0\psi(t,u,w)$. We are interested in the behaviour of $\lim_{t\to\infty} t^{-1}\Lambda_t(u,0)$.

Define the function $\chi : \mathbb{R} \to \mathbb{R}$ by $\chi(u) := \partial_w R(u,w)|_{w=0}$, assume that $\chi(0) < 0$ and $\chi(1) < 0$. Then

Lemma (Keller-Ressel, 2009)

There exist an interval $\mathcal{I} \subset \mathbb{R}$ and a unique function $w \in C(\mathcal{I}) \cap C^1(\mathcal{I}^\circ)$ such that $R(u, w(u)) = 0$, for all $u \in \mathcal{I}$ with $w(0) = w(1) = 0$. Define the set $\mathcal{J} := \{u \in \mathcal{I} : F(u, w(u)) < \infty\}$ and the function $h(u) := F(u, w(u))$ on $\mathcal{J}$, then

$$\lim_{t\to\infty} t^{-1}\Lambda_t(u,0) = \lim_{t\to\infty} t^{-1}\phi(t,u,0) = h(u), \quad \text{for all } u \in \mathcal{J},$$

$$\lim_{t\to\infty} \psi(t,u,0) = w(u), \quad \text{for all } u \in \mathcal{I}.$$

For convenience, we shall write $\Lambda_t(u)$ in place of $\Lambda_t(u,0)$.

Share measure

Let us define the share measure $\tilde{\mathbb{P}}(A) := \mathbb{E}\left((X_t - X_0)\,\mathbb{1}_A\right)$, and $\tilde{\mathcal{I}} := \{u : u + 1 \in \mathcal{I}\}$. Define $\tilde{h}(u) := \lim_{t\to\infty} t^{-1}\tilde{\Lambda}_t(u)$, then $\tilde{h}(u) = h(u+1)$, for all $u \in \tilde{\mathcal{J}}$ and $\tilde{h}^*(x) = h^*(x) - x$, for all $x \in \mathbb{R}$.

Lemma

h^* and $\tilde{h}^*$ are both good rate functions, strictly convex and admit zero as a unique minimum attained at $h'(0)$ and $h'(1)$ with $h'(0) < 0 < h'(1)$.

Theorem

The process $\left(t^{-1}(X_t - X_0)\right)_{t>0}$ satisfies a LDP as t tends to infinity under $\mathbb{P}$ (resp. $\tilde{\mathbb{P}}$) with the good rate function h^* (resp. $\tilde{h}^*$). Furthermore (likewise under $\tilde{\mathbb{P}}$),

$$-\lim_{t\to\infty} t^{-1}\log\mathbb{P}\left(\frac{X_t - x_0}{t} \in (a,b)\right) = \inf_{x\in(a,b)} h^*(x), \quad \text{for all } a < b.$$

A Black-Scholes intermezzo

Let us consider the Black-Scholes model: $\mathrm{d}S_t = \Sigma S_t \,\mathrm{d}W_t$, with $S_0 > 0$ and $\Sigma > 0$. We have the following

$$h_{\mathrm{BS}}(u, \Sigma) := \lim_{t\to\infty} t^{-1} \log \mathbb{E}\left(\mathrm{e}^{u(X_t - X_0)}\right) = \frac{1}{2} u(u-1)\Sigma^2, \qquad \text{for all } u \in \mathbb{R},$$

$$h^*_{\mathrm{BS}}(x) := \sup_{u\in\mathbb{R}} \{ux - h_{\mathrm{BS}}(u, \Sigma)\} = \left(x + \Sigma^2/2\right)^2 / \left(2\Sigma^2\right), \qquad \text{for all } x \in \mathbb{R}.$$

Lemma (Forde & Jacquier, 2009)

The process $\left(t^{-1}(X_t - X_0)\right)_{t>0}$ satisfies a LDP as t tends to infinity under $\mathbb{P}$ (resp. $\tilde{\mathbb{P}}$) with the good rate function h^*_{BS} (resp. $\tilde{h}^*_{\mathrm{BS}}$). Furthermore (likewise under $\tilde{\mathbb{P}}$),

$$-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{X_t - x_0}{t} \in (a, b)\right) = \inf_{x\in(a,b)} h^*_{\mathrm{BS}}(x)$$

$$= h^*_{\mathrm{BS}}(b, \Sigma)\, \mathbb{1}_{\{2b \le -\Sigma^2\}} + h^*_{\mathrm{BS}}(b, \Sigma)\, \mathbb{1}_{\{2a \le -\Sigma^2\}},$$

for all $a < b$ (here we have $h'(0) = -\Sigma^2/2$).

Final steps: from probabilities to implied volatility

Lemma

As t tends to infinity, we have the following option price asymptotics:

(*i*) $$-\lim_{t\to\infty} t^{-1}\log\mathbb{E}\left(S_t - S_0\mathrm{e}^{xt}\right)_+ = \tilde{h}^*(x), \quad \text{for all } x \geq h'(1),$$

(*ii*) $$-\lim_{t\to\infty} t^{-1}\log\left(S_0 - \mathbb{E}\left(S_t - S_0\mathrm{e}^{xt}\right)_+\right) = \tilde{h}^*(x), \quad \text{for all } h'(0) \leq x \leq h'(1),$$

(*iii*) $$-\lim_{t\to\infty} t^{-1}\log\left(\mathbb{E}\left(S_0\mathrm{e}^{xt} - S_t\right)_+\right) = \tilde{h}^*(x), \quad \text{for all } x \leq h'(0).$$

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Define the following function on $\mathbb{R}$:

$$\hat{\sigma}^2_\infty(x) := 2\left(2h^*(x) - x + \left(1\!\!1_{\{x\in(h'(0),h'(1))\}} - 1\!\!1_{\{x\notin(h'(0),h'(1))\}}\right) 2\sqrt{h^*(x)\left(h^*(x)-x\right)}\right).$$

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Theorem

The function $\hat{\sigma}_\infty$ is continuous and $\lim_{t\to\infty}\hat{\sigma}^2_t(x) = \hat{\sigma}^2_\infty(x)$, for all $x\in\mathbb{R}$.

Note that here $\hat{\sigma}_t(x)$ corresponds to a strike $K = S_0\exp(xt)$.

Small-time asymptotics

We are interested in determining

$$\lambda(u) := \lim_{t\to 0} t\Phi_t(u/t, 0) = \lim_{t\to 0}\Big(t\phi(t, u/t, 0) + v_0 t\psi(t, u/t, 0)\Big), \quad \text{for all } u \in \mathcal{D}_\lambda.$$

Let us define the Fenchel-Legendre transform $\lambda^* : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ of λ by

$$\lambda^*(x) := \sup_{u\in\mathbb{R}} \{ux - \lambda(u)\}, \quad \text{for all } x \in \mathbb{R}.$$

Theorem

The random variable $(X_t - X_0)_{t\geq 0}$ satisfies a LDP with rate λ^* as t tends to zero.

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Proposition

The small-time implied volatility reads

$$\sigma_0(x) := \lim_{t\to 0}\sigma_t(x) = \frac{|x|}{\sqrt{2\lambda^*(x)}} \in [0,\infty], \quad \text{for all } x \in \mathbb{R}^*.$$

Small-time for continuous affine SV models

Assume that the process has continuous paths, i.e. $\mu \equiv 0$ and $\mathrm{m} \equiv 0$. Define

$$\lambda_0(u) := \lim_{t \to 0} t\psi(t, u/t, 0), \quad \text{for all } u \in \mathcal{D}_{\lambda_0}.$$

Lemma

$$\lambda_0(u) = \alpha_{22}^{-1}\left(-\alpha_{12}u + \zeta u \tan\left(\zeta u/2 + \arctan(\alpha_{12}/\zeta)\right)\right) \quad \text{and} \quad \mathcal{D}_{\lambda_0} = (u_-, u_+),$$

where $u_\pm := \zeta^{-1}(\pm\pi - 2\arctan(\alpha_{12}/\zeta)) \in \mathbb{R}_\pm$ and $\zeta := \det(\alpha)^{1/2} > 0$. Therefore we obtain

$$\lambda(u) = \lambda_0(u) + a_{11}u^2/2.$$

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$$\lambda(u) = \lambda_0(u) + a_{11}u^2/2.$$

- Everything works fine when there are no jumps, and λ is known in closed-form.
- Jumps have to be chosen carefully: Nutz & Muhle-Karbe (2010), Roper (2009)

One-dimensional exponential Lévy processes

Let $(X_t)_t \geq 0$ be a Lévy process with triplet (σ, η, ν). The standard Lévy assumptions as well as the martingale condition impose $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (x^2 \wedge 1)\, \nu(\mathrm{d}x) < \infty, \quad \int_{|x|\geq 1} \mathrm{e}^x \nu(\mathrm{d}x) < \infty, \quad \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^x - 1 - x\mathbb{1}_{|x|\leq 1}\right) \nu(\mathrm{d}x) = -\eta.$$

Now, $\Phi_t(u, 0) = \exp(t\phi(u))$. Hence

$$F(u, 0) = \phi_X(u) \quad \text{and} \quad R(u, 0) = 0.$$

The condition $\chi(1) < 0$ is not satisfied. However we can work directly with F, and

$$h \equiv \phi, \quad \text{and} \quad \mathcal{D} = \{u \in \mathbb{R} : h(u) < \infty\}.$$

Example: VG(a, b, c).

$$h_{\mathrm{VG}}(u) = \left(\frac{ab}{(a-u)(b+u)}\right)^c, \quad \text{and} \quad \mathcal{D} = (a, b).$$

Heston with jumps I

Consider the Heston model

$$\mathrm{d}X_t = \left(\delta - \frac{1}{2}V_t\right)\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \quad X_0 = x_0 \in \mathbb{R},$$
$$\mathrm{d}V_t = \kappa\left(\theta - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\,\mathrm{d}Z_t, \quad V_0 = v_0 > 0,$$
$$\mathrm{d}\left\langle W, Z\right\rangle_t = \rho\mathrm{d}t,$$

where $J := (J_t)_{t\geq 0}$ is a pure-jump Lévy process independent of $(W_t)_{t\geq 0}$. Assume

$$\chi\left(1\right) = \rho\sigma - \kappa < 0$$

(see also Forde-Jacquier-Mijatović, Keller-Ressel, Andersen-Piterbarg). The logarithmic moment generating function of the Heston model with jumps reads

$$\log \mathbb{E}\left(\mathrm{e}^{u(X_t - x_0)}\right) = K_H\left(u, t\right) + \tilde{K}_J\left(u\right)t,$$

with $\tilde{K}_J\left(u\right) := K_J\left(u\right) - uK_J\left(1\right)$ to ensure the martingale property. In terms of the functions F and R, we have

$$F\left(u, w\right) = \kappa\theta w + \tilde{K}_J\left(u\right), \quad \text{and} \quad R\left(u, w\right) = \frac{u}{2}\left(u-1\right) + \frac{\xi^2}{2}w^2 - \kappa w + \rho\xi u w.$$

Heston with jumps II

We know that, for all $u \in \left[u_-^h, u_+^h\right]$

$$K_H^{\infty}(u) := \lim_{t \to \infty} t^{-1} K_H(u,t) = \frac{\kappa\theta}{\xi^2}\left(\kappa - \rho\xi u - \sqrt{(\kappa - \rho\xi u)^2 - \xi^2 u(u-1)}\right),$$

so that

$$h(u) := \lim_{t \to \infty} t^{-1} \Lambda_t(u) = K_H^{\infty}(u) + \tilde{K}_J(u), \quad \text{for all } u \in \left[u_-^h \vee u_-^J, u_+^h \wedge u_+^J\right].$$

and

$$h^*(x) = \sup_{u \in \left[u_-^h \vee u_-^J, u_+^h \wedge u_+^J\right]} \{ux - h(u)\}, \quad \text{for all } x \in \mathbb{R}.$$

Note that Heston without jumps corresponds to Gatheral's SVI parameterisation, ensuring its no-arbitrage for large maturities (see Gatheral & Jacquier, 2010):

$$\hat{\sigma}_{\infty}(x) = \frac{\omega_1}{2}\left(1 + \omega_2 \rho x + \sqrt{(\omega_2 x + \rho)^2 + 1 - \rho^2}\right), \quad \text{for all } x \in \mathbb{R}.$$

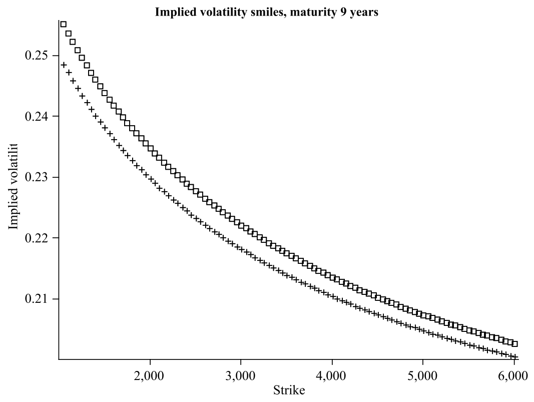
Heston with jumps III

Consider Normal Inverse Gaussian jumps, i.e.
J is an independent Normal Inverse Gaussian process with parameters $(\alpha, \beta, \mu, \delta)$ and Lévy exponent

$$K_{NIG}(u) = \mu u + \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2} \right).$$

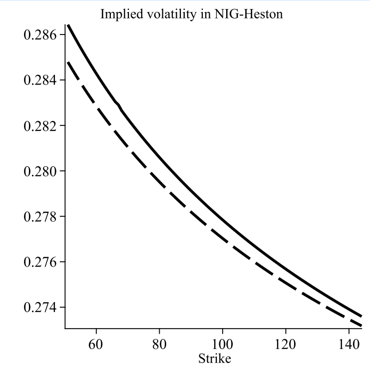
Then $u_{\pm}^{NIG} = -b \pm a$.

Numerical example: Heston without jumps



Heston (without jumps) calibrated on the Eurostoxx 50 on February, 15th, 2006, and then generated for $T = 9$ years. $\kappa = 1.7609$, $\theta = 0.0494$, $\sigma = 0.4086$, $v_0 = 0.0464$, $\rho = -0.5195$.

Numerical example: Heston with NIG jumps



We use the same parameters as before for Heston and the following for NIG: $\alpha = 7.104$, $\beta = -3.3$, $\delta = 0.193$ and $\mu = 0.092$. Heston (with jumps) calibrated on the Eurostoxx 50 on February, Note that, in the limit as $T \to \infty$, the smile Heston + NIG jumps exactly corresponds to a double Heston smile!!

Barndorff-Nielsen & Shephard (2001) I

$$\mathrm{d}X_t = -\left(\gamma k(\rho) + \frac{1}{2}V_t\right)\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t + \rho\,\mathrm{d}J_{\gamma t}, \quad X_0 = x_0 \in \mathbb{R},$$
$$\mathrm{d}V_t = -\gamma V_t \mathrm{d}t + \mathrm{d}J_{\gamma t}, \quad V_0 = v_0 > 0,$$

where $\gamma > 0$, $\rho < 0$ and $(J_t)_{t\geq 0}$ is a Lévy subordinator where the cgf of J_1 is given by $k(u) = \log \mathbb{E}\left(\mathrm{e}^{uJ_1}\right)$. $\mathcal{D}_\Lambda = (u_-, u_+)$, where

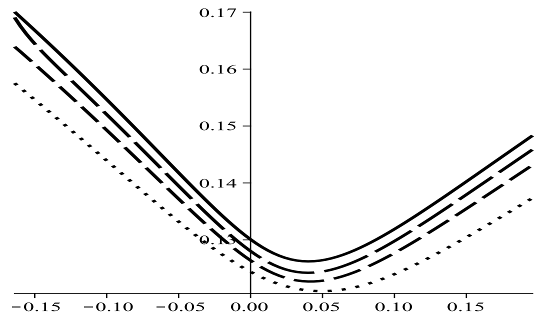
$$u_\pm := \frac{1}{2} - \rho\gamma \pm \sqrt{\frac{1}{4} - (2k^* - \rho)\gamma + \rho^2\gamma^2}.$$

with $k^* := \sup\{u > 0 : k(u) < \infty\}$. We deduce the two functions F and R,

$$R(u,0) = \frac{1}{2}\left(u^2 - u\right), \quad \text{and} \quad F(u,0) = \gamma k(\rho u) - u\gamma k(\rho).$$

Consider the Γ-BNS model, where the subordinator is $\Gamma(a,b)$-distributed with a, $b > 0$. Hence $k_\Gamma(u) = (b-u)^{-1}au$, and $u_\pm^\Gamma := \frac{1}{2} - \rho\gamma \pm \sqrt{\left(\frac{1}{2} - \rho\gamma\right)^2 + 2b\gamma} \in \mathbb{R}_\pm$.

Barndorff-Nielsen & Shephard II



Γ-BNS model with $a = 1.4338$, $b = 11.6641$, $v_0 = 0.0145$, $\gamma = 0.5783$, (Schoutens)
Solid line: asymptotic smile. Dotted and dashed: 5, 10 and 20 years generated smile.

Conclusion

Summary:

- Closed-form formulae for affine stochastic volatility models with jumps for large maturities.
- Closed-form formulae for continuous affine stochastic volatility models for small maturities.

Future research:

- Remove the conditions $\chi(0) < 0$ and $\chi(1) < 0$.
- What happens precisely in the small-time when jumps are added?
- Determine the higher-order correction terms (in t or t^{-1}).
- Statistical and numerical tests to assess the calibration efficiency.