Enhanced Convergence Results for Stochastic Tree Estimators

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Executive Summary

- Prove convergence of original stochastic tree estimators
  - weaker assumptions (first versus first-plus-epsilon moment); and
  - stronger mode of convergence (almost sure versus q-norm).
- Prove almost-sure convergence of bias-corrected stochastic tree estimators.

Valuation of American-style Options

- Valuation is done via dynamic programming through the recursive equations

$$H_k = \mathbb{E}\left[B_{k+1} \,|\, \mathcal{F}_k\right] \quad \text{and}$$
$$B_k = \max(H_k, P_k),$$

  where
  - H_k is the time-k hold value;
  - P_k is the time-k exercise value;
  - B_k is the time-k option value;
  - the terminal condition is $H_N = 0$;
  - N is option expiry; and
  - $k = k\Delta T$ denotes time.
- Note that we have suppressed the discount factor.

Stochastic Tree — Broadie and Glasserman 1997

- Brute-force valuation of the hold-value estimator.
- Let M be the *branching factor*.
- Given S_k generate M values of S_{k+1} (these are iid).
- Continue in this fashion for all k.
- $\mathbf{i} = (i_1, i_2, i_3, \ldots, i_N)$ denotes the path through the tree.
- Can specify exact location by **i** and depth k.

Stochastic Tree — Broadie and Glasserman 1997

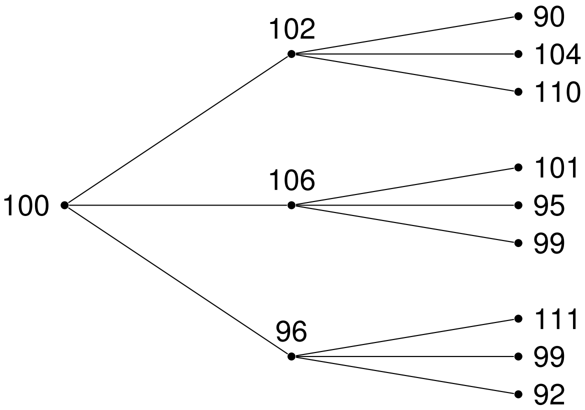

Figure: Two-period stochastic tree with branching factor of 3

High-biased Stochastic Tree Estimator

- A high-biased estimator uses the recursive equations

$$\tilde{H}^{\mathbf{i}}_{k,M} = \frac{1}{M} \sum_{i_{k+1}=1}^{M} \tilde{B}^{\mathbf{i}}_{k+1,M} \quad \text{and}$$

$$\tilde{B}^{\mathbf{i}}_{k,M} = \max(\tilde{H}^{\mathbf{i}}_{k,M}, P^{\mathbf{i}}_{k}),$$

where the terminal condition is $\tilde{H}^{\mathbf{i}}_{N,M} = 0$;

High-biased Estimator Consistency

- Assume that $\mathbb{E}[|P_k|^{q'}] < \infty$ for all k and for some $q' > 1$. Then the high-biased estimator converges in q-norm for any $0 < q < q'$ as $M \to \infty$.
(Theorem 1 of Broadie and Glasserman, 1997)
- Assumption — first-plus-epsilon absolute moment.
- Convergence — in q-norm.

Estimator Bias

- Define $\bar{H}^{\mathbf{i}}_{k,M} = \mathbb{E}[\tilde{H}^{\mathbf{i}}_{k,M}|\mathcal{F}_k]$
- Define the time-k bias as

$$\begin{aligned}\bar{H}^{\mathbf{i}}_{k,M} - H^{\mathbf{i}}_k &= \mathbb{E}[\tilde{B}^{\mathbf{i}}_{k+1,M} - B^{\mathbf{i}}_{k+1}|\mathcal{F}_k] \\ &= \mathbb{E}\big[\max(\tilde{H}^{\mathbf{i}}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(H^{\mathbf{i}}_{k+1}, P^{\mathbf{i}}_{k+1})|\mathcal{F}_k\big]\end{aligned}$$

- Add/subtract $\mathbb{E}[\max(\bar{H}^{\mathbf{i}}_{k+1,M}, P^{\mathbf{i}}_{k+1})|\mathcal{F}_k]$ gives

$$\begin{aligned}&\mathbb{E}\big[\max(\tilde{H}^{\mathbf{i}}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(\bar{H}^{\mathbf{i}}_{k+1,M}, P^{\mathbf{i}}_{k+1})|\mathcal{F}_k\big] \quad \text{(local)} \\ &\quad + \mathbb{E}\big[\max(\bar{H}^{\mathbf{i}}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(H^{\mathbf{i}}_{k+1}, P^{\mathbf{i}}_{k+1})|\mathcal{F}_k\big]. \quad \text{(global)}\end{aligned}$$

- We derive an approximation to the local bias.

Estimator Bias

- Let $1\!\!1_A = 1$ if A is true and $1\!\!1_A = 0$ otherwise.
- Note that

$$\mathbb{E}\left[1\!\!1_{\bar{H}^{\mathbf{i}}_{k+1,M} > P^{\mathbf{i}}_{k+1}} (\tilde{H}^{\mathbf{i}}_{k+1,M} - \bar{H}^{\mathbf{i}}_{k+1,M}) | \mathcal{F}_{k+1}\right] = 0$$

  and by nested expectations, so is the $\mathcal{F}_k$-conditional expectation.
- Subtract this term inside local bias to get

$$\mathbb{E}\big[1\!\!1_{\bar{H}^{\mathbf{i}}_{k+1,M} > P^{\mathbf{i}}_{k+1}} 1\!\!1_{\tilde{H}^{\mathbf{i}}_{k+1,M} \leq P^{\mathbf{i}}_{k+1}} (P^{\mathbf{i}}_{k+1} - \tilde{H}^{\mathbf{i}}_{k+1,M}) + 1\!\!1_{\bar{H}^{\mathbf{i}}_{k+1,M} \leq P^{\mathbf{i}}_{k+1}} 1\!\!1_{\tilde{H}^{\mathbf{i}}_{k+1,M} > P^{\mathbf{i}}_{k+1}} (\tilde{H}^{\mathbf{i}}_{k+1,M} - P^{\mathbf{i}}_{k+1}) | \mathcal{F}_k\big]$$

- Using Y's for $(H - P)$'s gives

$$\mathbb{E}\big[1\!\!1_{\bar{Y}^{\mathbf{i}}_{k+1,M} > 0} 1\!\!1_{\tilde{Y}^{\mathbf{i}}_{k+1,M} \leq 0} (-\tilde{Y}^{\mathbf{i}}_{k+1,M}) + 1\!\!1_{\bar{Y}^{\mathbf{i}}_{k+1,M} \leq 0} 1\!\!1_{\tilde{Y}^{\mathbf{i}}_{k+1,M} > 0} (\tilde{Y}^{\mathbf{i}}_{k+1,M}) \big| \mathcal{F}_k\big].$$

Time-$(k+1)$ Local Error in Hold Value Estimator

Reminder: $Y = H - P$

	Held: $\tilde{Y}^{\mathbf{i}}_{k+1,M} > 0$	Exercised: $\tilde{Y}^{\mathbf{i}}_{k+1,M} \leq 0$
Should Hold: $\bar{Y}^{\mathbf{i}}_{k+1,M} > 0$	0	$-\tilde{Y}^{\mathbf{i}}_{k+1,M}$
Should Exercise: $\bar{Y}^{\mathbf{i}}_{k+1,M} \leq 0$	$\tilde{Y}^{\mathbf{i}}_{k+1,M}$	0

Note that this error is always non-negative.

Approximation to Bias

- By CLT $\tilde{Y}^{\mathbf{i}}_{k+1,M} \sim N(\bar{Y}^{\mathbf{i}}_{k+1,M}, \bar{V}^{\mathbf{i}}_{k+1,M}/M)$ (approximately).
- Take $\tilde{Y}^{\mathbf{i}*}_{k+1,M} \sim N(\bar{Y}^{\mathbf{i}}_{k+1,M}, \bar{V}^{\mathbf{i}}_{k+1,M}/M)$ (exactly).
- Replace $\tilde{Y}^{\mathbf{i}}_{k+1,M}$ with $\tilde{Y}^{\mathbf{i}*}_{k+1,M}$ to get

$$\mathbb{E}\left[1\!\!1_{\bar{Y}^{\mathbf{i}}_{k+1,M}>0} 1\!\!1_{\tilde{Y}^{\mathbf{i}*}_{k+1,M}\le 0}(-\tilde{Y}^{\mathbf{i}*}_{k+1,M}) + 1\!\!1_{\bar{Y}^{\mathbf{i}}_{k+1,M}\le 0} 1\!\!1_{\tilde{Y}^{\mathbf{i}*}_{k+1,M}>0}(\tilde{Y}^{\mathbf{i}*}_{k+1,M}) \middle| \mathcal{F}_k \right]$$
$$= \int_0^\infty \int \int_D |\tilde{y}^*| \frac{1}{\sqrt{\bar{v}/M}} \phi\left(\frac{\tilde{y}^* - \bar{y}}{\sqrt{\bar{v}/M}} \right) f_{\bar{Y}^{\mathbf{i}}_{k+1,M}, \bar{V}^{\mathbf{i}}_{k+1,M} | \mathcal{F}_k}(\bar{y}, \bar{v})\, \mathrm{d}\tilde{y}^* \mathrm{d}\bar{y}\, \mathrm{d}\bar{v},$$

where $D = (0,\infty) \times (-\infty, 0] \cup (-\infty, 0] \times (0, \infty)$ and ϕ is the standard normal density function.

- Distributions of $\bar{Y}^{\mathbf{i}*}_{k+1,M}$ and $\tilde{Y}^{\mathbf{i}*}_{k+1,M}$ change at different rates with M.

Approximation to Bias

- Substitute $\bar{z} = \bar{y}\sqrt{M}$ and $\tilde{z}^* = \tilde{y}^*\sqrt{M}$ giving

$$\frac{1}{M}\int_0^\infty \int \int_D |\tilde{z}^*| \frac{1}{\sqrt{\bar{v}}}\phi\left(\frac{\tilde{z}^* - \bar{z}}{\sqrt{\bar{v}}}\right) f_{\bar{Y}^{\mathbf{i}}_{k+1,M}, \bar{V}^{\mathbf{i}}_{k+1,M}|\mathcal{F}_k}\left(\frac{\bar{z}}{\sqrt{M}}, \bar{v}\right) \mathrm{d}\tilde{z}^* \mathrm{d}\bar{z}\ \mathrm{d}\bar{v}.$$

- Convergence of $\tilde{Y}^{\mathbf{i}+*}_{k+1,M}$, $\bar{Y}^{\mathbf{i}+}_{k+1,M}$ and $\bar{V}^{\mathbf{i}+}_{k+1,M}$ to $Y^{\mathbf{i}}_{k+1}$, $Y^{\mathbf{i}}_{k+1}$ and $V^{\mathbf{i}}_{k+1}$ implies

$$\approx \frac{1}{M}\int_0^\infty \int \int_D |\tilde{z}^*| \frac{1}{\sqrt{\bar{v}}}\phi\left(\frac{\tilde{z}^* - \bar{z}}{\sqrt{\bar{v}}}\right) f_{\tilde{Y}^{\mathbf{i}*}_{k+1,M}, \bar{V}^{\mathbf{i}}_{k+1,M}|\mathcal{F}_k}\left(\frac{\tilde{z}^*}{\sqrt{M}}, \bar{v}\right) \mathrm{d}\tilde{z}^* \mathrm{d}\bar{z}\ \mathrm{d}\bar{v}.$$

- Undoing the $\tilde{z}^*$ and $\bar{z}$ substitutions gives

$$\int_0^\infty \int \int_D |\tilde{y}^*| \frac{1}{\sqrt{\bar{v}/M}}\phi\left(\frac{\tilde{y}^* - \bar{y}}{\sqrt{\bar{v}/M}}\right) f_{\tilde{Y}^{\mathbf{i}*}_{k+1,M}, \bar{V}^{\mathbf{i}}_{k+1,M}|\mathcal{F}_k}(\tilde{y}^*, \bar{v})\ \mathrm{d}\tilde{y}^* \mathrm{d}\bar{y}\ \mathrm{d}\bar{v}.$$

- Now integrate with respect to $\bar{y}$.

Approximation to Bias

- Local bias is approximately

$$\mathbb{E}\left[|\tilde{Y}^{\mathbf{i}*}_{k+1,M}| \, \Phi\left(\frac{-|\tilde{Y}^{\mathbf{i}*}_{k+1,M}|}{\sqrt{\bar{V}^{\mathbf{i}}_{k+1,M}/M}} \right) \Big| \mathcal{F}_k \right].$$

- Substitute $(\tilde{Y}^{\mathbf{i}}_{k+1,M}, \tilde{V}^{\mathbf{i}}_{k+1,M})$ for $(\tilde{Y}^{\mathbf{i}*}_{k+1,M}, \bar{V}^{\mathbf{i}}_{k+1,M})$ giving

$$\approx \mathbb{E}\left[|\tilde{Y}^{\mathbf{i}}_{k+1,M}| \, \Phi\left(\frac{-|\tilde{Y}^{\mathbf{i}}_{k+1,M}|}{\sqrt{\tilde{V}^{\mathbf{i}}_{k+1,M}/M}} \right) \Big| \mathcal{F}_k \right]$$

- Can be estimated in the simulation.

Bias-corrected Estimator

- Recursive equations for the corrected esitmator

$$\tilde{H}^{\mathbf{i}}_{k,M} = \frac{1}{M} \sum_{i_{k+1}=1}^{M} \tilde{B}^{\mathbf{i}}_{k+1,M} \quad \text{and}$$

$$\tilde{B}^{\mathbf{i}}_{k,M} = \max(\tilde{H}^{\mathbf{i}}_{k,M}, P^{\mathbf{i}}_{k}) - |\tilde{H}^{\mathbf{i}}_{k,M} - P^{\mathbf{i}}_{k}| \, \Phi\left(\frac{-|\tilde{H}^{\mathbf{i}}_{k,M} - P^{\mathbf{i}}_{k}|}{\sqrt{\tilde{V}^{\mathbf{i}}_{k,M}/M}}\right)$$

  where the terminal condition is $\tilde{H}^{\mathbf{i}}_{N,M} = 0$;
- Will now show this estimator converges almost surely.

Lemma (Bounds)

- Define the generic quantities

$$\tilde{U}^{\mathbf{i},p}_{k,M} = \frac{1}{M}\sum_{i_{k+1}=1}^{M} \cdots \frac{1}{M}\sum_{i_N=1}^{M} \max_{\tau\in[k,\ldots,N]} |P^{\mathbf{i}}_{\tau}|^p,$$

$$U^{\mathbf{i},p}_{k} = \mathrm{E}\Big[\max_{\tau\in[k,\ldots,N]} |P^{\mathbf{i}}_{\tau}|^p \Big| \mathcal{F}_k\Big],$$

These are almost surely finite if each $\mathbb{E}[|P^{\mathbf{i}}_k|^p] < \infty$.

Lemma (Bounds)

For all $\mathbf{i}$, $1 \leq p$, *and* k,

1. $|P^{\mathbf{i}}_k|^p \leq_1 U^{\mathbf{i},p}_k$
2. $|H^{\mathbf{i}}_k|^p \leq_1 U^{\mathbf{i},p}_k$ *and* $|\tilde{H}^{\mathbf{i}}_{k,M}|^p \leq \tilde{U}^{\mathbf{i},p}_{k,M}$,
3. $|B^{\mathbf{i}}_k|^p \leq_1 U^{\mathbf{i},p}_k$ *and* $|\tilde{B}^{\mathbf{i}}_{k,M}|^p \leq \tilde{U}^{\mathbf{i},p}_{k,M}$, *and*
4. $|V^{\mathbf{i}}_k|^p \leq_1 U^{\mathbf{i},2p}_k$ *and* $|\tilde{V}^{\mathbf{i}}_{k,M}|^p \leq (M/(M-1))^p \tilde{U}^{\mathbf{i},2p}_{k,M}$.

Lemma (Bounds Consistency) I

Lemma (Bounds consistency)

For all $\mathbf{i}$, $1 \leq q \leq p$, k, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if* $U_0^p < \infty$, *then*

1. $\tilde{U}_{k,M}^{\mathbf{i},q}$ *and* $U_k^{\mathbf{i},q}$ *are integrable,*
2. $\tilde{U}_{k,M}^{\mathbf{i},q} \rightarrow_1 U_k^{\mathbf{i},q}$ *and* $1/M \sum_{i_k=1}^{M} \tilde{U}_{k,M}^{\mathbf{i},q} \rightarrow_1 \mathrm{E}[U_k^{\mathbf{i},q} \| \mathcal{F}_{k-1}]$, *and*
3. $\mathrm{E}[\tilde{U}_{k,M}^{\mathbf{i},q} \| \mathcal{G}] =_1 \mathrm{E}[U_k^{\mathbf{i},q} \| \mathcal{G}]$.

- Consider arbitrary $\mathbf{i}$, $1 \leq q \leq p$, and k such that the lemma conditions are satisfied. As $|x|^q \leq |x|^p + 1$, integrability of $\tilde{U}_{k,M}^{\mathbf{i},q}$ and $U_k^{\mathbf{i},q}$ follows from integrability of $\tilde{U}_{k,M}^{\mathbf{i},p}$ and $U_k^{\mathbf{i},p}$. The latter is shown by taking the expected value of the definitions and expanding the domain of the max:

$$\mathrm{E}\left[\tilde{U}_{k,M}^{\mathbf{i},p}\right] = \mathrm{E}\left[U_k^{\mathbf{i},p}\right] = \mathrm{E}\left[\max_{\tau \in [k,\ldots,N]} |P_\tau^{\mathbf{i}}|^p\right] \leq \mathrm{E}\left[\max_{\tau \in [0,\ldots,N]} |P_\tau^{\mathbf{i}}|^p\right] = U_0^p < \infty.$$

Lemma (Bounds Consistency) II

Lemma (Bounds consistency)

For all $\mathbf{i}$, $1 \leq q \leq p$, k, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if* $U_0^p < \infty$, *then*

1. $\tilde{U}_{k,M}^{\mathbf{i},q}$ *and* $U_k^{\mathbf{i},q}$ *are integrable,*
2. $\tilde{U}_{k,M}^{\mathbf{i},q} \to_1 U_k^{\mathbf{i},q}$ *and* $1/M \sum_{i_k=1}^{M} \tilde{U}_{k,M}^{\mathbf{i},q} \to_1 \mathrm{E}[U_k^{\mathbf{i},q} \| \mathcal{F}_{k-1}]$, *and*
3. $\mathrm{E}[\tilde{U}_{k,M}^{\mathbf{i},q} \| \mathcal{G}] =_1 \mathrm{E}[U_k^{\mathbf{i},q} \| \mathcal{G}]$.

- The second part follows from the strong law of large numbers
- Third part an immediate consequence of taking the $\mathcal{G}$-conditional expectation.

Main Result: Theorem (Estimator Consistency) I

Theorem (Estimator consistency)

For all $\mathbf{i}$, $2 \leq p$, k, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if* $U_0^p < \infty$, *then*

1. $P_k^{\mathbf{i}}$, $H_k^{\mathbf{i}}$, $B_k^{\mathbf{i}}$, $V_k^{\mathbf{i}}$, $\tilde{H}_{k,M}^{\mathbf{i}+}$, $\tilde{B}_{k,M}^{\mathbf{i}+}$, *and* $\tilde{V}_{k,M}^{\mathbf{i}+}$ *are integrable,*
2. $\tilde{H}_{k,M}^{\mathbf{i}+} \to_1 H_k^{\mathbf{i}}$, $\tilde{B}_{k,M}^{\mathbf{i}+} \to_1 B_k^{\mathbf{i}}$, *and* $\tilde{V}_{k,M}^{\mathbf{i}+} \to_1 V_k^{\mathbf{i}}$, *and*
3. $\mathrm{E}[\tilde{H}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[H_k^{\mathbf{i}} \| \mathcal{G}]$, $\mathrm{E}[\tilde{B}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[B_k^{\mathbf{i}} \| \mathcal{G}]$, *and* $\mathrm{E}[\tilde{V}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[V_k^{\mathbf{i}} \| \mathcal{G}]$.

- Consider arbitrary $\mathbf{i}$, k, p, and $\mathcal{G} \subset \mathcal{F}_k$ such that the theorem conditions are satisfied. Integrability follows from the bounds established by Lemma (Bounds) and the integrability of $\tilde{U}_{k,M}^{\mathbf{i},1}$ and $\tilde{U}_{k,M}^{\mathbf{i},2}$ by Lemma (Bounds Consistency). The rest of the theorem trivially holds for N as $\tilde{B}_{k,N}^{\mathbf{i}} = B_N^{\mathbf{i}} = P_N^{\mathbf{i}}$.

Main Result: Theorem (Estimator Consistency) II

Theorem (Estimator consistency)

For all $\mathbf{i}$, $2 \leq p$, k, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if* $U_0^p < \infty$, *then*

1. $P_k^{\mathbf{i}}$, $H_k^{\mathbf{i}}$, $B_k^{\mathbf{i}}$, $V_k^{\mathbf{i}}$, $\tilde{H}_{k,M}^{\mathbf{i}+}$, $\tilde{B}_{k,M}^{\mathbf{i}+}$, *and* $\tilde{V}_{k,M}^{\mathbf{i}+}$ *are integrable,*
2. $\tilde{H}_{k,M}^{\mathbf{i}+} \to_1 H_k^{\mathbf{i}}$, $\tilde{B}_{k,M}^{\mathbf{i}+} \to_1 B_k^{\mathbf{i}}$, *and* $\tilde{V}_{k,M}^{\mathbf{i}+} \to_1 V_k^{\mathbf{i}}$, *and*
3. $\mathrm{E}[\tilde{H}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[H_k^{\mathbf{i}} \| \mathcal{G}]$, $\mathrm{E}[\tilde{B}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[B_k^{\mathbf{i}} \| \mathcal{G}]$, *and* $\mathrm{E}[\tilde{V}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[V_k^{\mathbf{i}} \| \mathcal{G}]$.

Can show that

$$\lim_M \tilde{H}_{k,M}^{\mathbf{i}} = \lim_M \frac{1}{M} \sum_{i_{k+1}=1}^{M} \tilde{B}_{k+1,M}^{\mathbf{i}} =_1 \mathrm{E}\left[\lim_M \tilde{B}_{k+1,M}^{\mathbf{i}} \middle\| \mathcal{F}_k\right]$$
$$=_1 \mathrm{E}\left[B_{k+1}^{\mathbf{i}} \| \mathcal{F}_k\right] = H_k^{\mathbf{i}}.$$

Likewise for $\tilde{V}_{k,M}^{\mathbf{i}}$ and $\tilde{B}_{k,M}^{\mathbf{i}}$

Theorem (Estimator consistency)

For all $\mathbf{i}$, $2 \leq p$, k, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if* $U_0^p < \infty$, *then*

1. $P_k^{\mathbf{i}}$, $H_k^{\mathbf{i}}$, $B_k^{\mathbf{i}}$, $V_k^{\mathbf{i}}$, $\tilde{H}_{k,M}^{\mathbf{i}+}$, $\tilde{B}_{k,M}^{\mathbf{i}+}$, *and* $\tilde{V}_{k,M}^{\mathbf{i}+}$ *are integrable,*
2. $\tilde{H}_{k,M}^{\mathbf{i}+} \to_1 H_k^{\mathbf{i}}$, $\tilde{B}_{k,M}^{\mathbf{i}+} \to_1 B_k^{\mathbf{i}}$, *and* $\tilde{V}_{k,M}^{\mathbf{i}+} \to_1 V_k^{\mathbf{i}}$, *and*
3. $\mathrm{E}[\tilde{H}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[H_k^{\mathbf{i}} \| \mathcal{G}]$, $\mathrm{E}[\tilde{B}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[B_k^{\mathbf{i}} \| \mathcal{G}]$, *and* $\mathrm{E}[\tilde{V}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[V_k^{\mathbf{i}} \| \mathcal{G}]$.

- The third part follows for k immediately from the second part, the bounds established by Lemma (Bounds), the consistency of those bounds as established by Lemma (Bounds Consistency), and another result.
- The entire theorem then holds for all k by induction.

Main Result II: Lemma (Uncorrected Estimator Consistency)

- Above result relies on the 2nd moment only because of the variance term in the corrected estimators.
- This result implies the almost sure convergence of the uncorrected estimators under the condition that $U_0^1 < \infty$ — existence of a first absolute moment.

Conclusion

- Prove convergence of original stochastic tree estimators
  - weaker assumptions (first versus first-plus-epsilon moment); and
  - stronger mode of convergence (almost sure versus q-norm).
- Prove almost-sure convergence of bias-corrected stochastic tree estimators.
- On to Davison's companion presentation