Spectral Decomposition of Option Prices in Fast Mean-Reverting Stochastic Volatility Models

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What is Spectral Theory?

Roughly speaking, generalization of eigenvalue equations for matrices to eigenvalue equations for linear operators

$$\begin{bmatrix} & | & \\ - & M & - \\ & | & \end{bmatrix} \begin{bmatrix} | \\ e_i \\ | \end{bmatrix} = \lambda_i \begin{bmatrix} | \\ e_i \\ | \end{bmatrix} \qquad \longrightarrow \qquad \mathcal{L}\psi_i = \lambda_i \psi_i$$

- ψ_i is eigenfunction corresponding to eigenvalue, λ_i
- The set of $\{\lambda_i\}$ for which the eigenvalue equation can be solved is the *spectrum* of $\mathcal{L}$

Q: Who Uses This Stuff Anyway?

A: Physicists!

$$\text{Heat equation:} \qquad \frac{\partial}{\partial t}u = \nabla^2 u$$

$$\text{Wave equation:} \qquad \frac{\partial^2}{\partial t^2}u = \nabla^2 u$$

$$\text{Schrödinger equation:} \qquad i\frac{\partial}{\partial t}u = Hu$$

Above PDE's can be separated into temporal and spatial components

The spatial component satisfies eigenvalue equation

Example: 1-D Heat Equation: $\partial_t u = \partial^2_{xx} u$

Try solution of the form $u(t, x) = g(t)\psi(x)$

$$g'\psi = g\psi'' \qquad \Rightarrow \qquad \frac{g'}{g} = \frac{\psi''}{\psi}$$

LHS function of t only, RHS function of x only

$\Rightarrow$ Both sides must equal constant, $-\lambda_i$

$$\psi_i'' = -\lambda_i \psi_i \qquad \qquad g_i' = -\lambda_i g_i$$

Spatial component solves eigenvalue equation with $\mathcal{L} = \partial^2_{xx}$

BC's fix the spectrum, $\{\lambda_i\}$

By linearity of heat equation, general solution is linear combination

$$u(t, x) = \sum_i A_i g_i(t) \psi_i(x)$$

Constants, $\{A_i\}$ determined by IC: $u(0, x) = f(x)$.

Q: Can Financial Mathematicians Use Spectral Analysis Too?

Let's see ... some recent work:

- Vadim Linetsky
  "Spectral Decomposition of the Option Value"
  "Exotic Spectra"

- Alan Lewis
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. . . and only if your last name begins with the letter "L"

My C.V.

Matthew J Lorig

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EDUCATION

B.S., Physics, *University of Minnesota* 2004

- Graduated Summa Cum Laude with Highest Distinction
- Barry Goldwater National Scholar

Ph.D., Physics, *UC - Santa Barbara* exp. 2011

- National Defense Science and Engineering Graduate Fellow

Graduating soon . . . Extended C.V. available upon request ☺

Q: Do Physicists Know Any Other Useful Math?

A: You bet! Example from Quantum Mechanics

If you can solve exactly: $H^0 \psi_n^0 = E_n^0 \psi_n^0$

But you can not solve: $(H^0 + \epsilon H^1)\psi_n^\epsilon = E_n^\epsilon \psi_n^\epsilon$

If ϵ small, try perturbation theory

$$E_n^\epsilon = E_n^0 + \epsilon E_n^1 + \epsilon^2 E_n^2 + \dots$$

$$\psi_n^\epsilon = \psi_n^0 + \epsilon \psi_n^1 + \epsilon^2 \psi_n^2 + \dots$$

Collect terms with like powers of ϵ

$\mathcal{O}(1)$	$H^0 \psi_n^0 = E_n^0 \psi_n^0$	(look familiar?)
$\mathcal{O}(\epsilon)$	$H^0 \psi_n^1 + H^1 \psi_n^0 = E_n^0 \psi_n^1 + E_n^1 \psi_n^0$	(often solvable)

Some Finance ... (finally)

Fast mean-reverting stochastic volatility models under $\widetilde{\mathbb{P}}$

$$dX_t = \left(r - \frac{1}{2} f^2(Y_t^\epsilon)\right) dt + f(Y_t^\epsilon) d\widetilde{W}_t,$$

$$dY_t^\epsilon = \left[\frac{1}{\epsilon}(m - Y_t^\epsilon) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \Lambda(Y_t^\epsilon)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} d\widetilde{B}_t,$$

$$d\left\langle \widetilde{W}, \widetilde{B} \right\rangle_t = \rho\, dt.$$

- $X_t = \log S_t$ with stochastic volatility $f(Y_t^\epsilon)$
- $\epsilon << 1 \Rightarrow Y_t^\epsilon$ fast mean-reverting
- $\Lambda(y)$ is market price of volatility risk
- $f(y)$ and $\Lambda(y)$ are bounded

Option Pricing

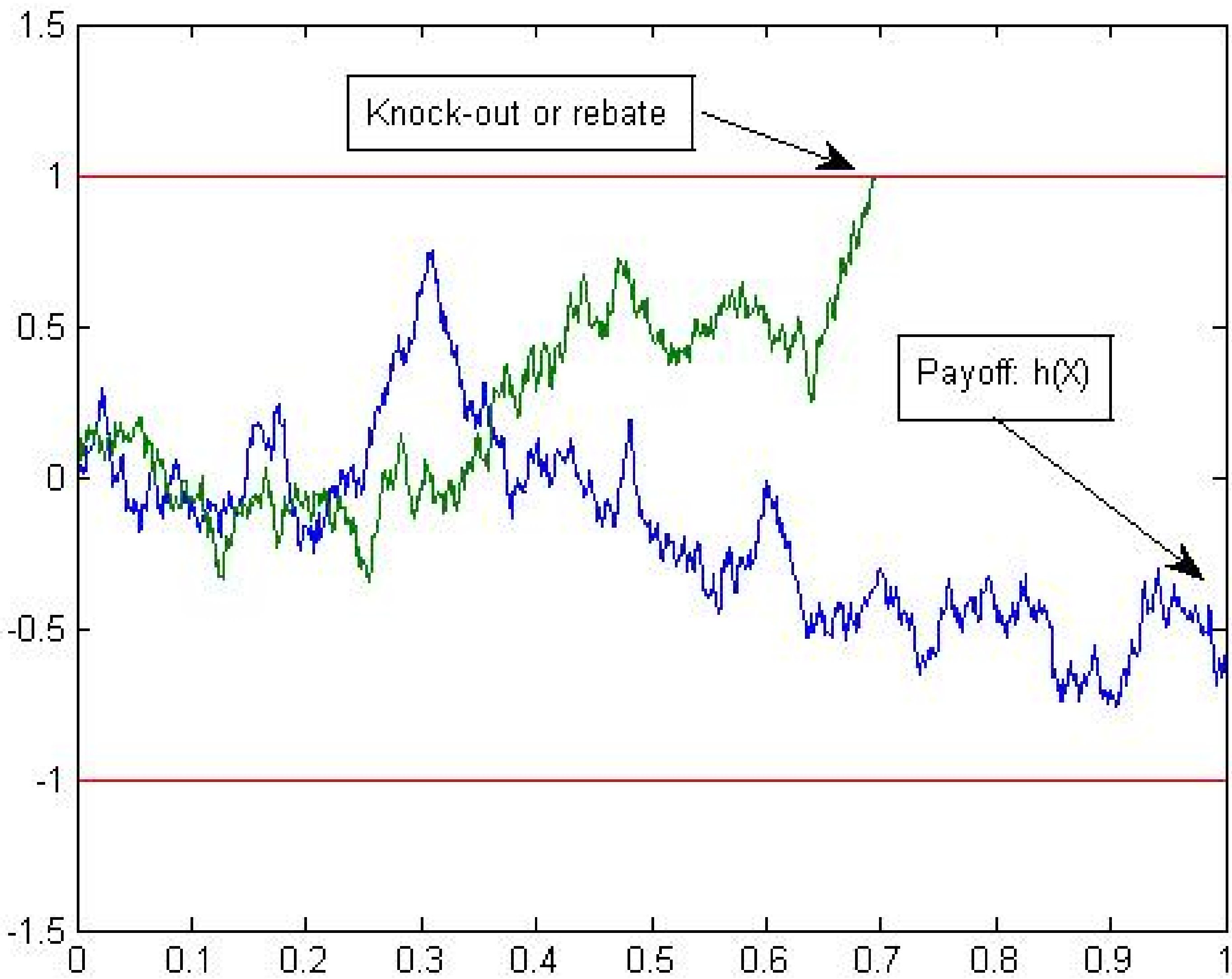
Consider option on X_t with payoff $h(X_\tau)$

$$\tau = \inf\{s > 0 : X_s \notin (x_l, x_u)\} \wedge T,$$

$$h : [x_l, x_u] \to \mathbb{R} \qquad -\infty \leq x_l < x_u \leq \infty$$

Examples of such options:

- European options (send $x_l \to -\infty$ and $x_u \to +\infty$)
- Single- and double-barrier knock-out options
- Rebate options



Knock-out or rebate
Payoff: h(X)
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Risk-Neutral Pricing

$$P_t^\epsilon = e^{rt}\,\widetilde{\mathbb{E}}\left[e^{-r\tau}h(X_\tau)\middle|\,X_t, Y_t^\epsilon\right] =: P^\epsilon(t, X_t, Y_t^\epsilon),$$

Feynman-Kac $\Rightarrow$ Option Pricing PDE

$$\begin{aligned}
\left(\partial_t - r + \mathcal{L}^\epsilon_{X,Y}\right) P^\epsilon &= 0, \\
P^\epsilon(T, x, y) &= h(x), \\
P^\epsilon(t, x_l, y) &= h(x_l) =: R_l, && \text{(not needed if } x_l = -\infty) \\
P^\epsilon(t, x_u, y) &= h(x_u) =: R_u. && \text{(not needed if } x_u = +\infty)
\end{aligned}$$

$\mathcal{L}^\epsilon_{X,Y}$ is infinitesimal generator of (X_t, Y_t)

Separation of Variables

Try $u^\epsilon(t,x,y) = g^\epsilon(t)\Psi^\epsilon(x,y)$ in $\left(\partial_t - r + \mathcal{L}^\epsilon_{X,Y}\right) u^\epsilon = 0$

$$\frac{-\left(\partial_t - r\right) g^\epsilon}{g^\epsilon} = \frac{\mathcal{L}^\epsilon_{X,Y}\,\Psi^\epsilon}{\Psi^\epsilon}$$

Both sides must be constant

$$-\left(\partial_t - r\right) g^\epsilon_q = \lambda^\epsilon_q g^\epsilon_q \qquad\qquad \mathcal{L}^\epsilon_{X,Y}\,\Psi^\epsilon_q = \lambda^\epsilon_q \Psi^\epsilon_q.$$

Candidate solutions are

$$P^\epsilon(t,x,y) = \Psi^\epsilon_r(x,y) + \int A^\epsilon_\omega g^\epsilon_\omega(t)\Psi^\epsilon_\omega(x,y)d\omega, \quad \text{(continuous spectrum)}$$

$$P^\epsilon(t,x,y) = \Psi^\epsilon_r(x,y) + \sum_m A^\epsilon_m g^\epsilon_m(t)\Psi^\epsilon_m(x,y). \qquad \text{(discrete spectrum)}$$

$\Psi^\epsilon_r(x,y)$ is steady-state solution, found by setting $\lambda_q = r$.

Boundary Conditions

A convenient B.C. for Temporal component is

$$g_q^\epsilon(T) = 1$$

Eigenfunctions satisfy

$$\Psi_q^\epsilon(x_l, y) = 0 \quad \text{if} \quad x_l > -\infty,$$
$$\Psi_q^\epsilon(x_u, y) = 0 \quad \text{if} \quad x_u < \infty.$$

Steady-state solution satisfies

$$\lim_{x \to x_l} \Psi_r^\epsilon(x, y) = R_l$$
$$\lim_{x \to x_u} \Psi_r^\epsilon(x, y) = R_u$$

R_l and R_u are rebates paid upon hitting x_l or x_u respectively

For given λ_q^ϵ, expression for $g_q^\epsilon(t)$ is easy to find

$$-\left(\partial_t - r\right)\, g_q^\epsilon = \lambda_q^\epsilon\, g_q^\epsilon$$

$$g_q^\epsilon(T) = 1$$

$$g_q^\epsilon(t) = \exp\left[(\lambda_q^\epsilon - r)(T-t)\right]$$

For general $f(y)$, no analytic expression for $\Psi_q^\epsilon(x,y)$

$$\mathcal{L}_{X,Y}^\epsilon \Psi_q^\epsilon = \lambda_q^\epsilon \Psi_q^\epsilon \qquad (+\text{ B.C.'s})$$

$$\begin{aligned}\mathcal{L}_{X,Y}^\epsilon = &\left[\left(r - \frac{1}{2}f^2(y)\right)\partial_x + \frac{1}{2}f^2(y)\partial_{xx}^2\right] \\ &+ \frac{1}{\sqrt{\epsilon}}\left[\rho\nu\sqrt{2}f(y)\partial_{xy}^2 - \nu\sqrt{2}\Lambda(y)\partial_y\right] \\ &+ \frac{1}{\epsilon}\left[(m-y)\,\partial_y + \nu^2\partial_{yy}^2\right].\end{aligned}$$

Try (singular) Perturbation Theory w.r.t. ϵ

$$\mathcal{L}^{\epsilon}_{X,Y} = \frac{1}{\epsilon}\mathcal{L}^{(-2)} + \frac{1}{\sqrt{\epsilon}}\mathcal{L}^{(-1)} + \mathcal{L}^{(0)}$$

$$\Psi^{\epsilon}_q = \Psi^{(0)}_q + \sqrt{\epsilon}\Psi^{(1)}_q + \epsilon\Psi^{(2)}_q + \dots,$$

$$\lambda^{\epsilon}_q = \lambda^{(0)}_q + \sqrt{\epsilon}\lambda^{(1)}_q + \epsilon\lambda^{(2)}_q + \dots$$

Since A^{ϵ}_q, g^{ϵ}_q and P^{ϵ} depend on ϵ we should expand them as well

$$A^{\epsilon}_q = A^{(0)}_q + \sqrt{\epsilon}A^{(1)}_q + \dots,$$

$$g^{\epsilon}_q = g^{(0)}_q + \sqrt{\epsilon}g^{(1)}_q + \dots,$$

$$P^{\epsilon} = P^{(0)} + \sqrt{\epsilon}P^{(1)} + \dots,$$

$$g^{(0)}_q(t) = e^{(\lambda^{(0)}_q - r)(T-t)},$$

$$g^{(1)}_q(t) = \lambda^{(1)}_q(T-t)g^{(0)}_q(t).$$

Insert expansions for Ψ_q^ϵ and λ_q^ϵ into $\mathcal{L}_{X,Y}^\epsilon \Psi_q^\epsilon = \lambda_q^\epsilon \Psi_q^\epsilon$

Find $\Psi_q^{(0)}$ and $\Psi_q^{(1)}$ can be written

$$\Psi_q^{(0)} = e^{cx}\psi_q^{(0)}(x) \qquad \text{no } y\text{-dependence}$$

$$\Psi_q^{(1)} = e^{cx}\psi_q^{(1)}(x) \qquad \text{no } y\text{-dependence}$$

where

$$c = \frac{-\left(r - \frac{1}{2}\overline{\sigma}^2\right)}{\overline{\sigma}^2}$$

$$\overline{\sigma}^2 = \left\langle f^2 \right\rangle$$

$$\langle v \rangle := \int v(y)\,dF_Y(y)$$

F_Y is distribution of $Y_\infty^\epsilon \sim N(m, \nu^2)$ under $\mathbb{P}$

$\left\{\psi_q^{(0)}, \lambda'^{(0)}_q\right\}$ **satisfy eigenvalue equation**

$$\partial^2_{xx}\psi_q^{(0)} = \lambda'^{(0)}_q \psi_q^{(0)} \qquad\qquad \lambda'^{(0)}_q = \frac{2\lambda_q^{(0)} + \overline{\sigma}^2 c^2}{\overline{\sigma}^2}$$

∂^2_{xx} is self-adjoint on inner product space $(u, v) := \int_{x_l}^{x_u} \bar{u}\, v\, dx$

$\Rightarrow \left\{\psi_q^{(0)}\right\}$ form orthonormal basis (we will use this later)

$\left\{\psi_q^{(1)}, \lambda_q^{(1)}\right\}$ **satisfy**

$$\left(V_2\chi + V_3\eta_q\right)\partial_x\psi_q^{(0)} + \left(V_2\xi_q + V_3\gamma_q\right)\psi_q^{(0)} = \frac{\overline{\sigma}^2}{2}\left({\lambda'}_q^{(0)} - \partial_{xx}^2\right)\psi_q^{(1)} + \lambda_q^{(1)}\psi_q^{(0)}$$

$$V_2 = \frac{\nu}{\sqrt{2}}\left\langle\Lambda\phi'\right\rangle,$$
$$V_3 = \frac{-\rho\nu}{\sqrt{2}}\left\langle f\phi'\right\rangle$$

ϕ satisfies $\mathcal{L}^{(-2)}\phi = f^2 - \overline{\sigma}^2$

$\chi, \eta_q, \xi_q, \gamma_q$ are functions of ${\lambda'}_q^{(0)}, c$

Slow Down Turbo! Can I get a Review?

$$P^{\epsilon}(t,x,y) = P^{(0)}(t,x) + \sqrt{\epsilon}\,P^{(1)}(t,x) + \ldots$$

$$P^{(0)}(t,x) = e^{cx}\left(\psi_r^{(0)} + \sum_m A_m^{(0)} g_m^{(0)} \psi_m^{(0)}\right)$$

$$P^{(1)}(t,x) = e^{cx}\left(\psi_r^{(1)} + \sum_m \left(A_m^{(1)} g_m^{(0)} \psi_m^{(0)} + A_m^{(0)} g_m^{(1)} \psi_m^{(0)} + A_m^{(0)} g_m^{(0)} \psi_m^{(1)}\right)\right)$$

- $\left\{\psi_m^{(0)}, {\lambda'}_m^{(0)}\right\}$ solve: $\partial_{xx}^2 \psi_m^{(0)} = {\lambda'}_m^{(0)} \psi_m^{(0)}$
- $\left\{\psi_m^{(0)}\right\}$ orthonormal
- $\left\{\psi_m^{(1)}, \lambda_m^{(1)}\right\}$ satisfy equation in terms of $\left\{\psi_m^{(0)}, \lambda_m^{(0)}\right\}$
- Have expressions for $g_m^{(0)}$ and $g_m^{(1)}$ in terms of $\lambda_m^{(0)}$ and $\lambda_m^{(1)}$

(Continuous spectrum $\sum \to \int$)

What about $A_m^{(0)}$ and $A_m^{(1)}$?

Use $\underline{P^{(0)}(T,x) = h(x)}$ and $\underline{g_m^{(0)}(T) = 1}$

$$h(x) = e^{cx}\left(\psi_r^{(0)}(x) + \sum_m A_m^{(0)}\psi_m^{(0)}(x)\right)$$

$$\left(\psi_n^{(0)}, e^{-cx}h - \psi_r^{(0)}\right) = \sum_m A_m^{(0)}\left(\psi_n^{(0)}, \psi_m^{(0)}\right)$$

$$= \sum_m A_m^{(0)}\delta_{m,n} \qquad \text{(from orthonormality)}$$

$$A_m^{(0)} = \left(e^{-cx}h - \psi_r^{(0)}, \psi_m^{(0)}\right)$$

Similar calculation with $\underline{P^{(1)}(T,x) = 0}$ and $\underline{g_m^{(0)}(T) = 0}$ gives

$$A_m^{(1)} = -\left(\psi_m^{(0)}(x), \psi_r^{(1)}(x)\right) - \sum_n A_n^{(0)}\left(\psi_m^{(0)}(x), \psi_n^{(1)}(x)\right)$$

Example: European Option

Step 1: Find Expressions for $\psi_r^{(0)}(x)$ and $\psi_r^{(1)}(x)$

$$\lim_{x\to\pm\infty} \psi_r^{(i)}(x) = 0 \quad i = 0, 1$$

$$\partial_{xx}^2 \psi_r^{(0)} = {\lambda'}_r^{(0)} \psi_r^{(0)},$$

$$\frac{\overline{\sigma}^2}{2}\left({\lambda'}_r^{(0)} - \partial_{xx}^2\right)\psi_r^{(1)} = \left(V_2\chi + V_3\eta_r\right)\partial_x\psi_r^{(0)} + \left(V_2\xi_r + V_3\gamma_r\right)\psi_r^{(0)}.$$

$$\psi_r^{(0)}(x) = \psi_r^{(1)}(x) = 0$$

Step 2: Find Expressions for $\psi_\omega^{(0)}(x)$ and $\lambda_\omega^{(0)}$

$$\partial_{xx}^2 \psi_\omega^{(0)} = {\lambda'}_\omega^{(0)} \psi_\omega^{(0)}$$

$$\psi_\omega^{(0)}(x) = \frac{1}{\sqrt{2\pi}} e^{i\omega x},$$

$${\lambda'}_\omega^{(0)} = -\omega^2, \qquad \lambda_\omega^{(0)} = -\frac{1}{2}\left(\overline{\sigma}^2 c^2 + \overline{\sigma}^2 \omega^2\right).$$

Step 3: Find Expressions for $\psi_\omega^{(1)}(x)$ and $\lambda_\omega^{(1)}$

$$
\begin{aligned}
&(V_2\chi + V_3\eta_\omega)\,\partial_x\psi_\omega^{(0)} + (V_2\xi_\omega + V_3\gamma_\omega)\,\psi_\omega^{(0)} \\
&\qquad = \frac{\overline{\sigma}^2}{2}\left(\lambda'^{(0)}_\omega - \partial^2_{xx}\right)\psi_\omega^{(1)} + \lambda_\omega^{(1)}\psi_\omega^{(0)}
\end{aligned}
$$

$$
\begin{aligned}
\psi_\omega^{(1)}(x) &= \psi_\omega^{(0)}(x), \\
\lambda_\omega^{(1)} &= V_3\,\beta_\omega + V_2\,\zeta_\omega, \\
\beta_\omega &= (c+i\omega)^3 - (c+i\omega)^2, \\
\zeta_\omega &= (c+i\omega)^2 - (c+i\omega).
\end{aligned}
$$

Step 4: Find Expressions for $A_\omega^{(0)}$ and $A_\omega^{(1)}$

$\psi_r^{(0)} = \psi_r^{(1)} = 0$, so expressions for $A_\omega^{(0)}$ and $A_\omega^{(1)}$ simplify

$$A_\omega^{(0)} = \left(\psi_\omega^{(0)}, e^{-cx} h\right)$$

$$\begin{aligned} A_\omega^{(1)} &= -\int A_\nu^{(0)} \left(\psi_\omega^{(0)}, \psi_n^{(1)}\right) d\nu \\ &= -\int A_\nu^{(0)} \delta\left(\omega - \nu\right) d\nu \\ &= -A_\omega^{(0)} \end{aligned}$$

Step 5: Expressions for $P^{(0)}(t,x)$ and $P^{(1)}(t,x)$

$$P^{(0)}(t,x) = e^{cx} \int A_\omega^{(0)} \underline{g_\omega^{(0)}(t)} \psi_\omega^{(0)}(x) d\omega$$

$$\begin{aligned} P^{(1)}(t,x) &= e^{cx} \int \Big(A_\omega^{(1)} g_\omega^{(0)}(t) \psi_\omega^{(0)}(x) + A_\omega^{(0)} g_\omega^{(1)}(t) \psi_\omega^{(0)}(x) \\ &\qquad + A_\omega^{(0)} g_\omega^{(0)}(t) \psi_\omega^{(1)}(x) \Big) d\omega \\ &= e^{cx} \int \Big(\cancel{-A_\omega^{(0)} g_\omega^{(0)}(t) \psi_\omega^{(0)}(x)} + A_\omega^{(0)} g_\omega^{(1)}(t) \psi_\omega^{(0)}(x) \\ &\qquad \cancel{+ A_\omega^{(0)} g_\omega^{(0)}(t) \psi_\omega^{(0)}(x)} \Big) d\omega \\ &= e^{cx} \int A_\omega^{(0)} \underline{g_\omega^{(1)}(t)} \psi_\omega^{(0)}(x) d\omega \end{aligned}$$

Group Parameters V_2^ϵ and V_3^ϵ

- Adding fast mean-reverting factor of volatility adds
  - two unknown functions (f, Λ)
  - five unobservable parameters $(\epsilon, m, \nu, \rho, y)$

  to Black-Scholes framework

- Knowledge of these functions and parameters not needed to give approximate price of option

- P_0 corresponds to Black-Scholes price of an option with $\sigma \to \overline{\sigma}$

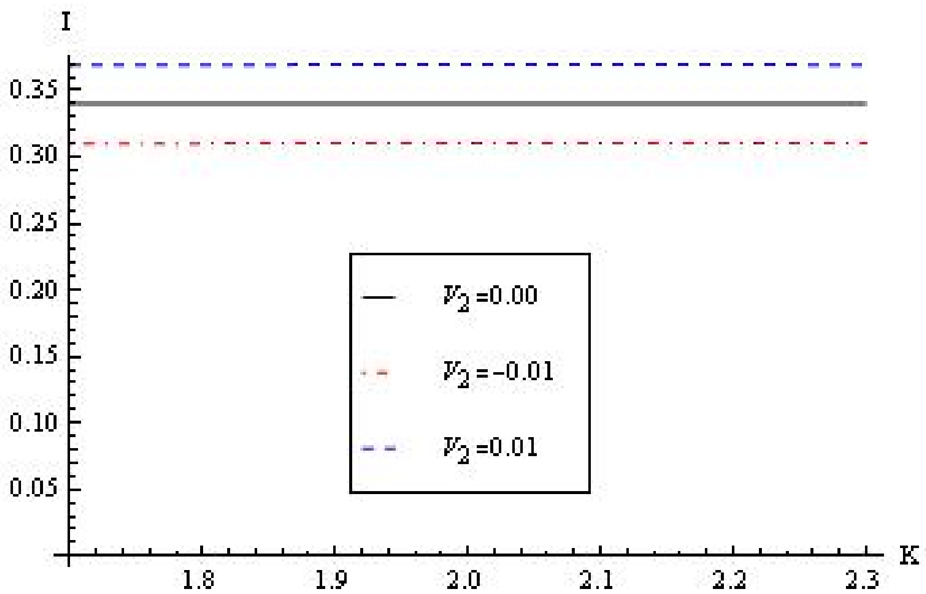
- $\sqrt{\epsilon} P_1$ is linear in group parameters V_2^ϵ and V_3^ϵ

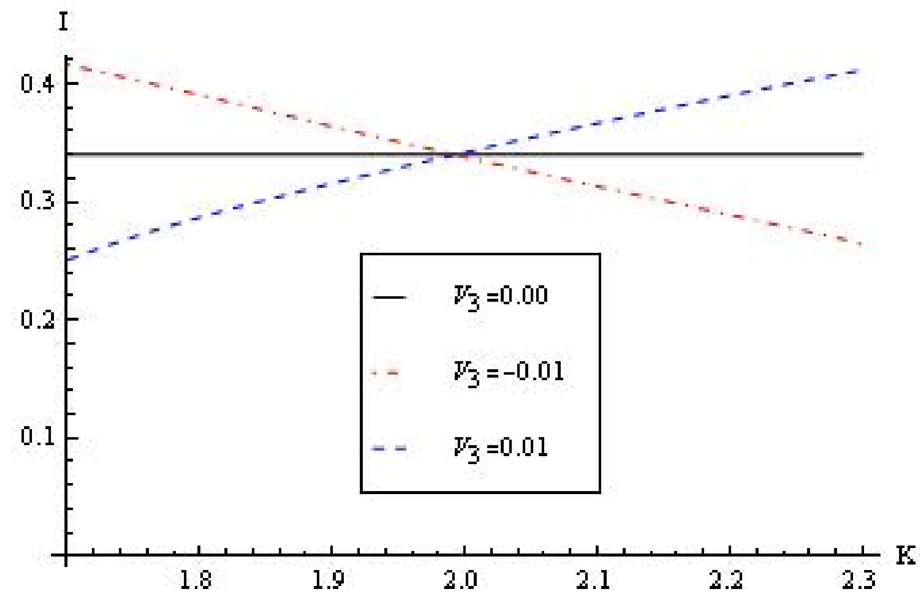
$$V_2^\epsilon = \sqrt{\epsilon}\frac{\nu}{\sqrt{2}} \langle \Lambda \phi' \rangle , \qquad V_3^\epsilon = -\sqrt{\epsilon}\frac{\rho\nu}{\sqrt{2}} \langle f \phi' \rangle$$

- True for all options in this framework — not just European

Effect of V_2^ϵ and V_3^ϵ on Implied Volatility

$$S_t = 2.0,\ T - t = 0.5$$



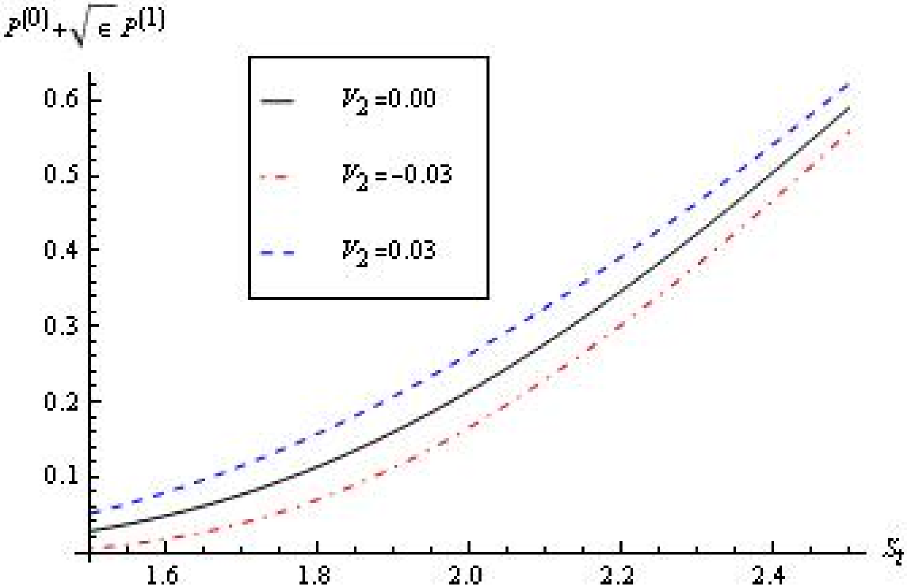


V_2^ϵ controls overall level of vol.

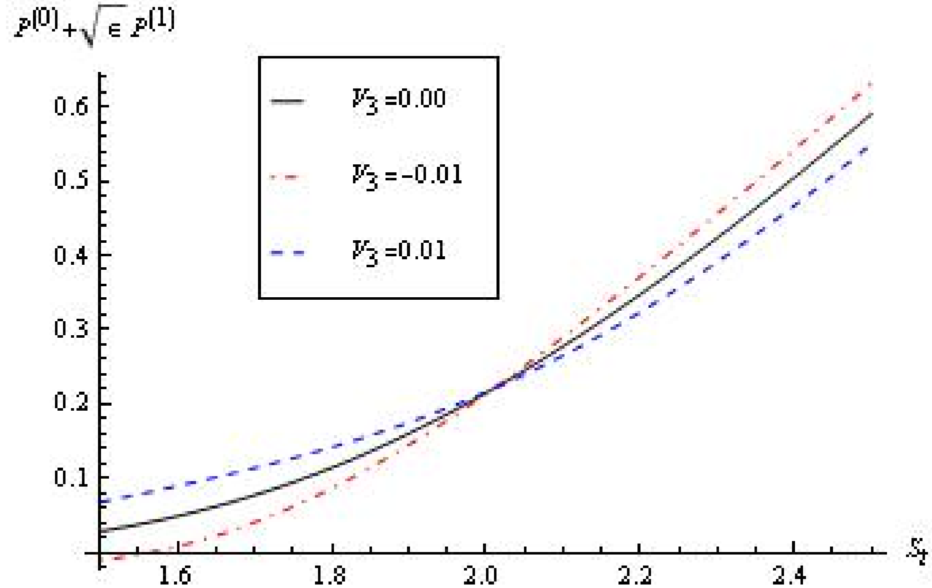
V_3^ϵ controls ATM skew.

European Call Price vs S_t

$K = 2.0$, $T - t = 0.5$



Effect of V_2^ϵ.



Effect of V_3^ϵ.

Example: Double-Barrier Knock-Out Option

Step 1: Find Expressions for $\psi_r^{(0)}(x)$ and $\psi_r^{(1)}(x)$

Option "knocks-out" at $X_t = x_l$ and $X_t = x_u \Rightarrow R_l = R_u = 0$

$$\lim_{x \to x_l} \psi_r^{(i)}(x) = 0 \quad i = 0, 1$$

$$\lim_{x \to x_u} \psi_r^{(i)}(x) = 0 \quad i = 0, 1$$

$$\partial_{xx}^2 \psi_r^{(0)} = {\lambda'}_r^{(0)} \psi_r^{(0)},$$

$$\frac{\overline{\sigma}^2}{2} \left({\lambda'}_r^{(0)} - \partial_{xx}^2 \right) \psi_r^{(1)} = \left(V_2 \chi + V_3 \eta_r \right) \partial_x \psi_r^{(0)} + \left(V_2 \xi_r + V_3 \gamma_r \right) \psi_r^{(0)}.$$

Obvious solution is

$$\psi_r^{(0)}(x) = \psi_r^{(1)}(x) = 0$$

Step 2: Find Expressions for $\psi_m^{(0)}(x)$ and $\lambda_m^{(0)}$

$$
\begin{aligned}
\partial^2_{xx}\psi_m^{(0)} &= {\lambda'}_m^{(0)}\psi_m^{(0)},\\
\psi_m^{(0)}(x_l) &= 0,\\
\psi_m^{(0)}(x_u) &= 0.
\end{aligned}
$$

One can easily verify the following set of solutions

$$
\begin{aligned}
\psi_m^{(0)}(x) &= \sqrt{\frac{2}{x_u - x_l}}\sin\left(\alpha_m(x - x_l)\right), & \alpha_m &= \frac{m\pi}{x_u - x_l}\\
{\lambda'}_m^{(0)} &= -\alpha_m^2 & \lambda_m^{(0)} &= -\frac{1}{2}\left(\overline{\sigma}^2 c^2 + \overline{\sigma}^2\alpha_m^2\right).
\end{aligned}
$$

Step 3: Find Expressions for $\psi_m^{(1)}(x)$ and $\lambda_m^{(1)}$

$$0 = \psi_m^{(1)}(x_l) = \psi_m^{(1)}(x_u)$$

$$\begin{aligned}(V_2\chi + V_3\eta_m)\,\partial_x\psi_m^{(0)} + (V_2\xi_m + V_3\gamma_m)\,\psi_m^{(0)} \\ = \frac{\overline{\sigma}^2}{2}\left(\lambda'^{(0)}_m - \partial^2_{xx}\right)\psi_m^{(1)} + \lambda_m^{(1)}\psi_m^{(0)}\end{aligned}$$

$\left\{\psi_m^{(0)}\right\}$ form complete orthonormal basis on $L^2(x_l, x_u)$

Try: $$\psi_m^{(1)}(x) = \sum_{n\neq m} a_{m,n}^{(1)}\psi_n^{(0)}$$

Find: $$a_{m,n}^{(1)} = \frac{(V_2\chi + V_3\eta_m)\left(\psi_n^{(0)}, \partial_x\psi_m^{(0)}\right)}{\lambda_m^{(0)} - \lambda_n^{(0)}}$$

$$\lambda_m^{(1)} = V_2\xi_m + V_3\gamma_m$$

Step 4: Find Expressions for $A_m^{(0)}$ and $A_m^{(1)}$

$\psi_r^{(0)} = \psi_r^{(1)} = 0$, so expressions for $A_m^{(0)}$ and $A_m^{(1)}$ simplify

$$A_m^{(0)} = \left(\psi_m^{(0)}, e^{-cx} h\right)$$

$$\begin{aligned} A_m^{(1)} &= -\sum_n A_n^{(0)} \left(\psi_m^{(0)}, \psi_n^{(1)}\right) \\ &= -\sum_n A_n^{(0)} \sum_k a_{n,k}^{(1)} \left(\psi_m^{(0)}, \psi_k^{(1)}\right) \\ &= -\sum_n A_n^{(0)} \sum_k a_{n,k}^{(1)} \delta_{m,k} \\ &= -\sum_n A_n^{(0)} a_{n,m}^{(1)} \end{aligned}$$

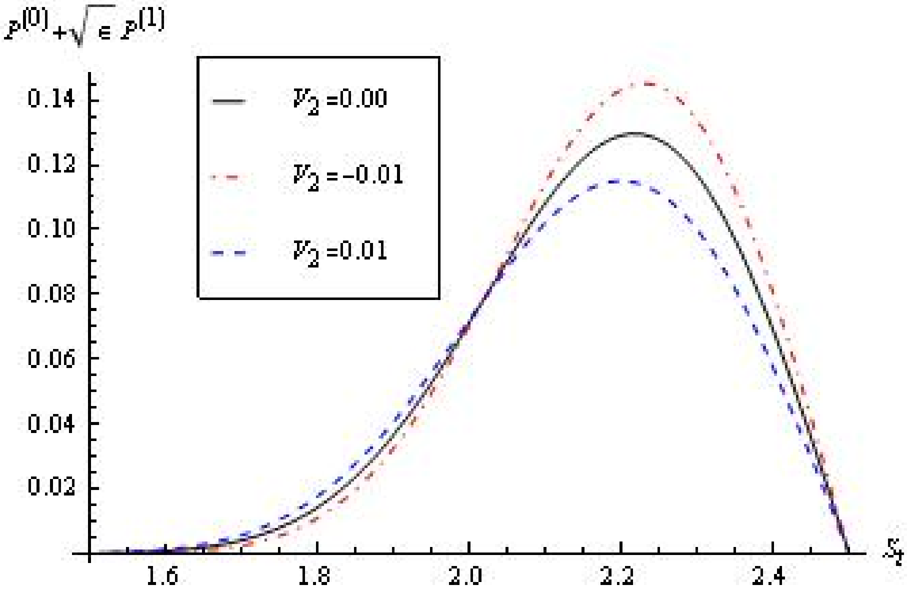
Step 5: Expressions for $P^{(0)}(t,x)$ and $P^{(1)}(t,x)$

$$P^{(0)}(t,x) = e^{cx} \sum_m A_m^{(0)} g_m^{(0)}(t) \psi_m^{(0)}(x),$$

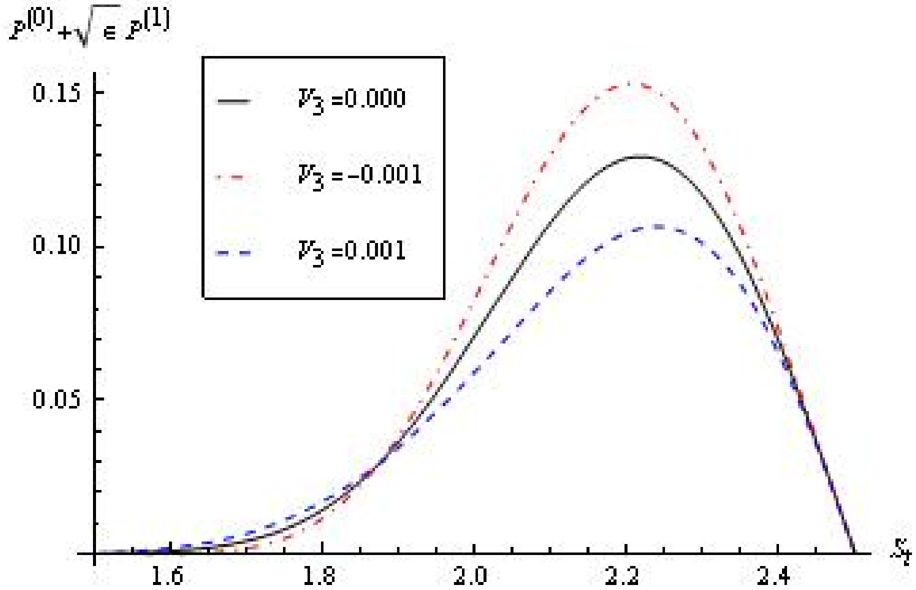
$$\begin{aligned} P^{(1)}(t,x) &= e^{cx} \sum_m \Big(A_m^{(1)} g_m^{(0)}(t) \psi_m^{(0)}(x) + A_m^{(0)} g_m^{(1)}(t) \psi_m^{(0)}(x) \\ &\qquad + A_m^{(0)} g_m^{(0)}(t) \psi_m^{(1)}(x) \Big) \\ &= e^{cx} \sum_m A_m^{(0)} g_m^{(1)}(t) \psi_m^{(0)}(x) \\ &\qquad + e^{cx} \sum_m \sum_n g_m^{(0)}(t) \left(A_m^{(0)} a_{m,n}^{(1)} \psi_n^{(0)}(x) - A_n^{(0)} a_{n,m}^{(1)} \psi_m^{(0)}(x) \right), \end{aligned}$$

Double-Barrier Call Price vs S_t

$S_l = 1.5,\ K = 2.0,\ S_u = 2.5,\ T - t = 0.5$



Effect of V_2^ϵ.



Effect of V_3^ϵ.

Some Key Advantages of Spectral Approach

- Fast convergence: $P \sim \sum e^{\lambda_n t}$, $\lambda_n \sim -n^2$
- Simple Implementation: Just solve few eigenvalue equations
- Widely Applicable: Works for European, Barrier, Rebate Options
- Only need two new parameters $(V_2^\epsilon, V_3^\epsilon)$ to give approximate price of options.

Thank You!