Option pricing by Recursive Projection

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Getting Started

- Goal of our project: build precise and fast algorithms to price a large class of complex derivatives;
- Type of complexity: the value of the asset depends on events that take place in successive moments in time;
- Example: Bermudan options, discrete dividends for American options (increasing the number of potential exercise dates)
- Allow pricing of contingent claims written on bundles of several assets, or that depend on several variables, e.g. price and volatility for stochastic volatility models:
  - Jump-diffusion;
  - Levy processes.

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Goals

- Definition of the problem;
- Intuition underlying the recursive projection;
- Application of recursive projection to option pricing;
- Application to Bermudan options;
- Application to discrete dividends.

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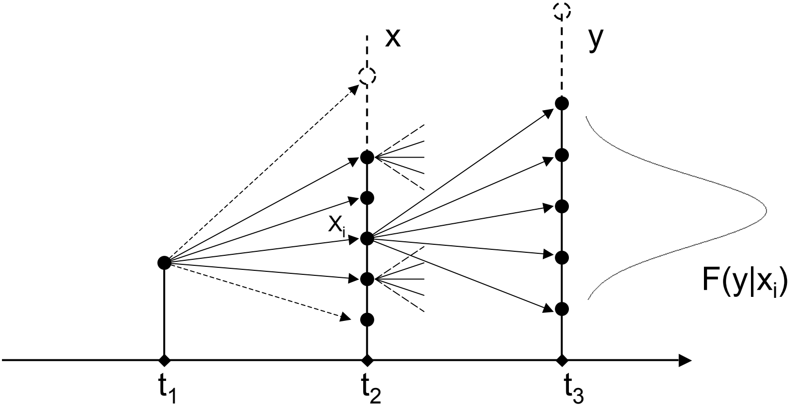
Example: 2 Steps



$$V(s, t_1) \longleftarrow V(X, t_2) \longleftarrow H(Y, t_3)$$

x : value of the underlying, $H(x, t)$: payoff function, $V(x, t)$: value function

How much?

- How much are we willing to pay for this contract?
- In the European case, the price of the option is the (risk neutral) expectation of the future cash flow:

$$
\begin{aligned}
V(x,t) &= \mathbb{E}^{Q}\left\{ e^{-r(T-t)} H(y) \,\middle|\, \mathscr{F}_t \right\} \\
&= \int_{-\infty}^{\infty} G(t,T;x,y) H(y)\, dy,
\end{aligned}
$$

y : value of the underlying at T,

$H(y)$: payoff of the option at T.

- We have to make some assumptions on how the underlying asset evolves.

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Pricing as a linear operator

- Consider a Riemann sum equivalent of the integral

$$V(x,t) = \int H(y,T)G(t,T;x,y)dy \sim \sum_{j=1}^{N-1} H(\xi_j,T)G(t,T;x,\xi_j)\Delta y;$$

- Project the payoff function on an orthogonal basis e_i:

$$\int_{-\infty}^{\infty} G(T-t;x-y)\sum_i a_i(T)e_i(y)\,dy =$$

$$= \sum_i G(x,T-t)_i a_i(T).$$

We substituted again an integral with a summation.

- Sampling of functions can be seen as a form of functional projection, in our case on a localized base.

Integrals and projections

- We can compute the *price* of the option, or the *value* of the option as function of $x = \ln(S_t)$:

$$a(t)_{[1\times 1]} = G_x(T-t)_{[1\times n]} \cdot A(T)_{[n\times 1]},$$

  we disentangled the time and the space component of the problem.
- For m different values of x, we obtain a transition matrix:

$$A(t)_{[m\times 1]} = \mathbf{G}(X, T-t)_{[m\times n]} \cdot A(T)_{[n\times 1]},$$

X : vector $(x_1, \ldots, x_m)$ of *conditioning* values of the transition density.

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Heston model

- What if

$$\int_{-\infty}^{\infty} G(t,T;x,y)H(y)\,dy = \sum_i a_i(T)\int_{-\infty}^{\infty} G(t,T;x,y)e_i(y)\,dy =$$

$$\sum_i \frac{a_i(T)}{2\pi}\int_{-\infty}^{\infty} e^{-iky}e_i(y)\,dy\int_{-\infty}^{\infty}\hat{G}(t,T;x,k)dk =$$

$$\sum_i \frac{a_i(T)}{2\pi}\int_{-\infty}^{\infty}\hat{G}(t,T;x,k)\hat{e}_i(-k)dk.$$

- Choosing the appropriate (flexible) basis function makes the inner product easy enough; for instance, using a numerical routine.
- So that again

$$A(t)_{[m\times 1]} = \mathbf{G}(X, T-t)_{[m\times n]}\cdot A(T)_{[n\times 1]},$$

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Bermudan Options

- Bermudan options are options that can be exercised at - usually equally spaced - fixed times before maturity $\{t_1, t_2, \ldots, t_i, t_{i+1}, \ldots, t_n\}$;
- Example : swaptions.
- Even for very simple dynamics of the underlying asset, like B&S, PDE's (let's keep it simple: a tree) have to be used, and intrinsic and time values have to be compared at every exercice date.
- Still, the dynamics between exercice dates is always the same, can we take advantage of this translational invariancy?

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Bermudan Options

- The matrix form of the linear operator:

$$A(t) = \mathbf{G}(X, T - t) \cdot A(T),$$

- $A(T)$ and $A(t)$ allow us to build the shape of the value function, and there are no constraints on these coefficients.
- Any kind of function can be send back in time.
- At each t_n continuation value $a_i(t)$ and intrinsic value $h_i(x, t_n) = (X_i - K)_+$ are compared.
- As long as $T - t$ is constant, the matrix $\mathbf{G}(T - t)$ is fixed.

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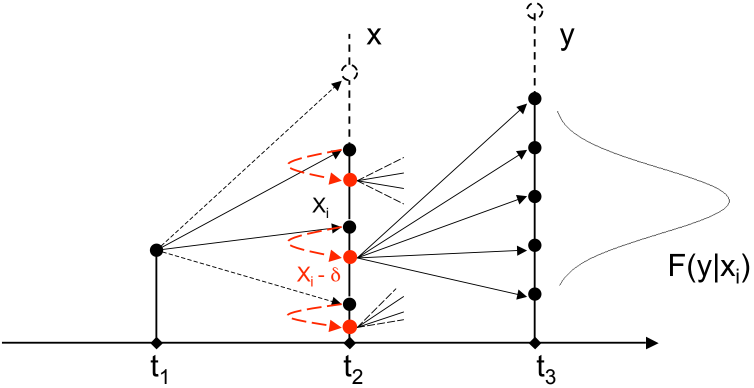
Intermediate Cash Flows



$$V(S, t_1) \longleftarrow \underbrace{\max\left((X-K)_+, \mathbf{G}(X-\delta(x), t_3-t_2)A(t_3)\right)}_{A(t_2)} \longleftarrow H(Y, t_3)$$

$\delta(x)$ can be whatever function.

Bermudan Put, Heston model

- Using the notation

$$dX = \left(r - \frac{1}{2}v(t)\right)dt + \sqrt{v(t)} \cdot dW_1$$
$$dv(t) = \left(a - bv(t)\right)dt + \alpha\sqrt{v(t)} \cdot dW_2.$$

- Parameters of the simulation

K	100	r	0
v_0	0.04	T	10 (τ =1)
ρ	(= $\langle dW_1, dW_2 \rangle$) 0.0	α	0.2
b	2	a	0.08

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Bermudan Put, Heston model

Recursive projection

S_t	80	100	120	
J		*Prices*		*sec*
8	33.601	24.761	18.337	*0.2*
9	33.600	24.759	18.334	*0.8*
10	33.600	24.758	18.333	*2.0*
True	33.608	24.760	18.341	
		errors (bp)		
8	2	0	2	
9	2	0	4	
10	2	1	4	

Value function sampled at $n = 2^J$ points

Finite-Difference (FD)

S_t	80.143	100	120.893	
L_T		*Prices*		*sec*
20	33.529	24.731	18.092	*13*
50	33.529	24.751	18.091	*30*
100	33.529	24.751	18.090	*60*
200	33.529	24.751	18.090	*121*
True	33.535	24.760	18.100	
		errors (bp)		
20	2	12	5	
50	2	4	5	
100	2	4	5	
200	2	4	5	

Space discretization parameter $m_s = 400$,
time discretization parameter L_T

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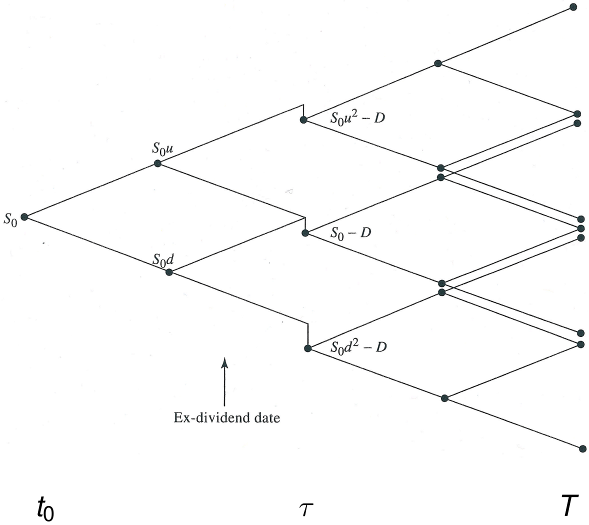
Dividends



Recombining tree :

$\#\text{ knots} = \frac{N(N+1)}{2}$

Non Recombining:

$\sim N/2^{\frac{N/2(N/2+1)}{2}}$

$$V(S, t_0) \longleftarrow \underbrace{\max\big((X-K)_+, \mathbf{G}(X-D, T-\tau)A(T)\big)}_{A(\tau)} \longleftarrow H(Y, T)$$

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American Call with dividend, BS model

$S_0 = 100, K = 100, \sigma = 0.2, r = 0, T = 3, \tau_1 = 1, \tau_2 = 2, d = 2$

Recursive Projection

J	*Price*	*sec*
6	12.144284	0.001
7	12.121988	0.002
8	12.120374	0.007
9	12.120875	0.03
True	12.1205	
	errors (bp)	
6	20	
7	1	
8	0.1	
9	0.3	

Binomial Tree

N	*Price*	*sec*
200	12.095077	0.2
500	12.115218	3
1000	12.116011	44
2000	12.119153	687
True (10000)	12.1205	(>6 days)
	errors (bp)	
200	21	
500	4	
1000	4	
2000	1	

Price with constant dividend yield $y = 0.02$: 12.075062 ($\sim 40bp$)

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1000	12.116011	44
2[illegible]	[illegible]19153	687
True (10000)	12.1205	(>6 days)

x 1000

x 100

errors (bp)

N	
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1000	4
2000	1

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Conclusions + Todos

- *Why does it work?*
  - Functional transforms are obtained by simple sampling of the relevant functions
  - Only variables *actually* appearing in the payoff contribute to the dimensionality of the problem
- Other forms of functional projection (faster convergence)
- Reduction of dimensionality: projections matrices are sparse, nonzero coefficients are all around the strike. Reduces computation in high dimension
- Happy of what we have seen so far: simple algorithms, fast and accurate implementation

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Theoretical Background

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- Duffie, D., Pan J. and K. Singleton (2000): “Transform analysis and asset pricing for affine jump-diffusions”, Econometrica
- Andricopoulos, A., Widdicks, M., Duck, P. and D. Newton (2003): “Universal option pricing using quadrature”, Journal of Financial Economics (extension to a multi asset framework in 2007)
- Hansen, L. P. and J. A. Scheinkman (2009) : “Long-term risk: an operator approach", Econometrica

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Bermudan Put, Heston model

Finite Difference (ADI)		
S_t	100	
Nb. Spatial Steps	*Prices*	*time (sec)*
200	24.725	*4*
300	24.744	*12*
400	24.751	*30*
800	24.758	*486*
True Price	24.760	
200	14	
300	6	
400	4	
800	1	

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