Exercise 5. (a) *Assume G is an infinite group. Prove that there is an infinite $A \subseteq G$ such that for all $\{x, y, z\} \subseteq A$ we have $xy = z$ if and only if $x = y = z$ and x is an idempotent.*

(b) *Does (a) remain true if G is assumed to be a semigroup instead of a group? Justify your answer.*

Proof. (a) Fix a well-order $<$ of G. Let $S = \{\sigma | \sigma \colon \{1,2,3\} \to \{1,2,3\}\}$ for each $\sigma \in S$ define $f_\sigma \colon [G]^3 \to \{0,1\}$ by $f_\sigma(\{x_1, x_2, x_3\}) = 1$ if $x_{\sigma(1)} x_{\sigma(2)} = x_{\sigma(3)}$, and 0 otherwise. Fix σ. We claim that there are no infinite 1-homogeneous sets for f_σ. Assume A is 1-homogeneous and infinite. If σ is constant, then $x^2 = x$ for all $x \in A$, which is impossible since G is a group. Thus the range of σ has at least two different values, and there is $i \le 3$ such that $\sigma(i) \neq \sigma(j)$ for $j \neq i$. Pick $\{y_1, y_2, y_3, y_4\}_<$ included in A. If $i = 3$, then $y_1 y_2 = y_3 = y_4$, hence $y_3 = y_4$, contradiction. If $i = 2$, then $y_1 y_2 = y_4$ and $y_1 y_3 = y_4$, hence $y_2 = y_3$, contradiction. Finally, if $i = 1$, then $y_1 = y_3 y_4 = y_2$, contradiction.

Therefore by Ramsey's theorem there is an infinite $A \subseteq G$ that is f_σ-0-homogeneous for all σ. Such A is as required.

(b) No. Take the semigroup $(\mathbb{N}, \min)$. Then $xy \in \{x, y\}$ for all x and y in $\mathbb{N}$. □

Exercise 6. *Assume $\{x_n\}_{n=1}^\infty$ is a sequence in $[0,1]$. Find a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that both $\lim_{k\to\infty} x_{n_k}$ and $\lim_{k\to\infty} x_{n_{k+1} - n_k}$ exist.*

Proof. Define a partition $f \colon [\mathbb{N}]^5 \to \{0,1,2\}$ by $f(\{n_1, n_2, n_3, n_4, n_5\}) = 1$, if $|x_{n_4 - n_5} - x_{n_2 - n_3}| \ge 1/n_1$, $f(\{n_1, n_2, n_3, n_4, n_5\}) = 2$ if $|x_{n_2} - x_{n_3}| \ge 1/n_1$, and 0 otherwise. A 0-homogeneous set gives a sequence as required, and there are no infinite 1-homogeneous or 2-homogeneous sets. □

Maps $f \colon \mathbb{N}^d \to Y$ and $g \colon \mathbb{N}^d \to Y$ are *isomorphic on A* (for some $A \in [\mathbb{N}]^\omega$) if for all $\{s, t\} \subseteq A^d$ we have $f(s) = f(t)$ if and only if $g(s) = g(t)$. A family $\mathcal{B}$ of functions forms a *basis* for $\mathcal{H}_d = \{f : \mathrm{dom}(f) = \mathbb{N}^d\}$ if for every $f \in \mathcal{H}_d$ there exist $g \in \mathcal{B}$ and $A \in [\mathbb{N}]^\omega$ such that f is isomorphic to g on A.

Exercise 7. (1) *Prove that for every $d \in \mathbb{N}$ there is a finite basis $\mathcal{B}_d$ for $\mathcal{H}_d$.*

(2) *Find the minimal possible size of $\mathcal{B}_d$ for $d = 2, 3, \ldots$.*

Proof. Let us first do this under the assumption that Y is finite. More precisely, $\mathcal{H}_d^0 = \{f \colon \mathbb{N}^d \to \mathbb{N} |$ the range of f is finite$\}$. For $s = \langle s_0, \ldots, s_{d-1} \rangle \in \mathbb{N}^d$ define $\preceq_s$ to be the relation on $d = \{0, \ldots, d-1\}$ defined by $i \preceq_s j$ if and only if $s_i \le s_j$. This is a quasi-ordering (i.e., it is transitive and reflexive, but not necessarily antisymmetric). Also, for all m and n less than d we have $m \preceq_s n$ or $n \preceq_s m$. Let the *height* of $\preceq_s$, $\mathrm{ht}(\preceq_s)$, be the size of the set $a_s = \{s_i | i < d\}$.

Let $\mathcal{S}_d = \{\preceq_s \colon s \in \mathbb{N}^d\}$. Note that $|\mathcal{S}_d| \le 2^{d^2}$, in particular it is finite. (For example, $\mathcal{S}_2$ has three elements, corresponding to $\{\langle m, n \rangle : m < n\}$, $\{\langle m, n \rangle : m > n\}$ and $\{\langle m, n \rangle : m = n\}$.) Let $\mathcal{E}_d$ be the set of all equivalence relations on $\mathcal{S}_d$; again, $\mathcal{E}_d$ is finite.

Claim 8. *For $p \in \mathcal{S}_d$, if $m = \mathrm{ht}(p)$ and $a \in [\mathbb{N}]^m$, then there is the unique $s = s(a, p) \in \mathbb{N}^d$ such that $\preceq_s = p$ and $\{s_i : i < d\} = a$. Moreover, $a_s = a$ and $\preceq_s = p$.* □

Define $\mathcal{B}_d^0 = \{f \colon \mathbb{N}^d \to \mathcal{P}(\mathcal{S}_d) :$ there is $\sim \in \mathcal{E}_d$ such that $f(s) = [s]_\sim\}$ (here $[s]_\sim$ stands for the $\sim$-equivalence class of s). Clearly, $\mathcal{B}_d^0$ is finite, and every function f such that there is $\sim$ in $\mathcal{E}_d$ for which $f(s) = f(t)$ if and only if $s \sim t$ is equivalent to a function in $\mathcal{B}_d^0$.

Claim 9. *$\mathcal{B}_d^0$ forms a basis for $\mathcal{H}_d^0$.*

Proof. Fix $f \in \mathcal{H}_d^0$, let $Y = \mathrm{range}(f)$. For $p \in \mathcal{S}_d$ define $f_p \colon [\mathbb{N}]^{\mathrm{ht}(p)} \to Y$ by

$$f_p(a) = f(s(a, p)).$$

Assume $A \subseteq \mathbb{N}$ is f_p-homogeneous, say $f_p(a) = i$ for all $a \in [A]^m$. Then $f(s) = f_p(a_s) = i$ for all $s \in A^d$ such that $\preceq_s = p$. Enumerate $\mathcal{S}_d$ and apply Ramsey's theorem for each $p \in \mathcal{S}_d$, to obtain $A \in [\mathbb{N}]^\omega$ and i_p ($p \in \mathcal{S}_d$) such that $f(s) = i_{\preceq_s}$ for all $s \in A^d$. Let $\sim \in \mathcal{E}_d$ be defined by $p \sim q$ if and only if $i_p = i_q$. Then $f \restriction A^d$ is isomorphic to the element of $\mathcal{B}_d^0$ given by $\sim$. □

It only remains to check that $\mathcal{B}_d^0$ is indeed the minimal basis for $\mathcal{H}_d^0$. But it is easily seen that if $f \in \mathcal{B}_d^0$, then the restriction of f to A^d for any infinite is equivalent to f. Thus if $\mathcal{B} \subseteq \mathcal{H}_d^0$ and $|\mathcal{B}| < |\mathcal{B}_d^0|$, and $f \in \mathcal{B}_d^0$ is such that no $g \in \mathcal{B}$ is equivalent to f, it follows that the restriction of f to A^d is not isomorphic to any $g \in \mathcal{B}$. Thus $\mathcal{B}$ cannot be a basis for $\mathcal{H}_d^0$.

Thus $|\mathcal{B}_d^0|$ is equal to the number of equivalence relations on the set $\mathcal{S}_d$, in particular $|\mathcal{B}_2^0| = 7$ since $|\mathcal{S}_2| = 3$. Also, $|\mathcal{S}_3| = 13$, so $|\mathcal{B}_3^0|$ is equal to the number of equivalence relations on a 13-element set.

Now the case when the range of f is not necessarily finite.[1] For $s = \langle n_1, \ldots, n_d \rangle \in \mathbb{N}^d$ and $X \subseteq \mathrm{ht}(\preceq_s)$ write $s \restriction X = \{m_i : i \in X\}$, where $\{m_1, \ldots, m_{\mathrm{ht}(\preceq_s)}\}$ is the increasing enumeration of $\{n_1, \ldots, n_d\}$.

Let $\mathcal{B}_d$ be the family of all functions f with domain $\mathbb{N}^d$ such that:

[1] I have intended to assign as the homework only the case when the range of f is finite; but now that I had this typo in the handout, here is an outline of the solution for the general case.

(1) for every $p \in \mathcal{S}_d$ there is $X^f_p \subseteq \mathrm{ht}(p)$,

(2) there is an equivalence relation $\sim^f$ on $\mathcal{S}_d$, such that $p \sim^f q$ implies $|X_p| = |X_q|$,

and $f(s) = \langle s \restriction X_{\preceq_s}, [s]_{\sim^f}\rangle$. (Thus, $f(s) = f(t)$ if and only if $s \restriction X_{\preceq_s} = t \restriction X_{\preceq_t}$ and $\preceq_s \sim \preceq_t$.)

Fix f. By using the Erdös–Rado theorem instead of Ramsey's theorem, find an infinite $A \subseteq \mathbb{N}$ such that f_p is canonical on $[\mathbb{N}]^{\mathrm{ht}(p)}$, for every $p \in \mathcal{S}_d$. That is, find an $X_p \subseteq \mathrm{ht}(p)$ such that $f_p(a) = f_p(b)$ if and only if $a \restriction X_p = b \restriction X_p$.

The problem is that we may have $f_p(a) = f_q(b)$ for distinct p and q, and we need to shrink A further in order to control the way in which this can happen. Define the function h_p on $[A]^{|X_p|}$ by

$$h_p(a \restriction X) = f_p(a)$$

(it is well-defined because f_p is canonical). Fix distinct p and q in $\mathcal{S}_d$. Let $m_{pq} = |X_p| + |X_q|$ and let $\mathcal{F}_{pq}$ be the family of all relations $R \subseteq [m_{pq}]^{|X_p|} \times [m_{pq}]^{|X_q|}$. Define $h_{pq} \colon [A]^{|X_p|+|X_q|} \to \mathcal{F}$ by $h_{pq}(\{n_1, \dots, n_{m_{pq}}\}_<) = R$ where R is the relation defined by

$(a, b) \in R$ if and only if $h_p(\{n_i : i \in a\}) = h_q(\{n_j : j \in b\})$.

Find an infinite $B \subseteq A$ and R such that h_{pq} has constant value R on $[B]^{m_{pq}}$. If $B = \{n_i : i \in \mathbb{N}\}$ is an increasing enumeration, then consider $B' = \{n_{2i} : i \in \mathbb{N}\}$ and $(B')^d$.

Claim 10. *There are only two possiblities for R: $R = \emptyset$ or R is the identity relation, the latter only in the case when $X_p = X_q$.*

Proof. Using the fact that both h_p and h_q are injections. Assume otherwise. There are s and t in $(B')^d$ such that $\preceq_s = p$, $\preceq_t = q$, $f(s) = f(t)$ and $s \restriction X_p \neq t \restriction X_q$. Let m be the minimal element of $X_p \Delta X_q$ such that the m-th element, i_s, of $(s \restriction X_p)$ differs from the m-th element, i_t, of $(t \restriction X_q)$. We may assume $i_s < i_t$. If $i_t = n_{2j}$ for some j, let $t' = t \setminus \{n_{2j}\} \cup \{n_{2j-1}\}$. Then $t' \in B^d$ and, by the homogeneity, $f(s) = f(t')$. Thus $f(t) = f(t')$. But $t \restriction X_q \neq t' \restriction X_q$, a contradiction. □

Repeat the construction of Claim for all pairs p, q in $\mathcal{S}_d$. At the end we have an infinite $C \subseteq \mathbb{N}$ such that $f \restriction C^d$ is isomorphic to some g in $\mathcal{B}^0_d$. □

Exercise 11. *Assume that $k \in \mathbb{N}$ and I_n is a set of size k for every $n \in \mathbb{N}$. Prove that there is an infinite $A \subseteq \mathbb{N}$ and a finite R such that $I_m \cap I_n = R$ for all m and n in A.*

Proof. By well-ordering $\bigcup_n I_n$ in type ω, we may assume $I_n \subseteq \mathbb{N}$ for all n. Let $I_n = \{l^n_1, \dots, l^n_k\}$, enumerated in increasing order. Define $f \colon [\mathbb{N}]^2 \to$ by $f(\{m, n\}_<) = 0$, if $l^m_i = l^n_j$ implies $i = j$ for all i, j, and $f(\{m, n\}_<) = \langle i, j\rangle$ if $l^m_i = l^n_j$ for some $i \neq j$. (Pick the lexicographically minimal such pair i, j.) If there is an infinite $\langle i, j\rangle$-homogeneous set for some i, j, pick $\{m, n, p\}_<$ in this set. Then $l^n_j = l^m_i = l^p_j = l^n_i$, contradicting the assumption that $l^n_i \neq l^n_j$ if $i \neq j$. By Ramsey's theorem, there is an infinite 0-homogeneous set, as required. □

Exercise 12. *Prove or disprove the following strengthening of Diagonalization Lemma:*

If each $\mathcal{F}_n \subseteq [\mathbb{N}]^\omega$ ($n \in \mathbb{N}$) is dense, then there is $A \in [\mathbb{N}]^\omega$ such that $A/n \in \bigcap_{m \leq n} \mathcal{F}_m$ for every $n \in \mathbb{N}$.

Proof. Let $\mathcal{F}_n = \{A : n + 1 \notin A\}$. If $A/n \in \mathcal{F}_n$ for all n, then $A = \emptyset$. Thus the strengthening is false. □

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